ANALYSIS OF RAPID BETATRON RESONANCE CROSSING

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Abstract

The reduction of transverse and longitudinal cooling requirements, the increased number of beam circulations, and the reduced cost, as compared to RLAs, are arguments to adopt the linear-field FFAG as the acceleration stage of a Neutrino Factory[1]. Because of the large range of central momenta, $\pm 50\%$ $\delta p/p$, and negative uncorrected chromaticity, the non-scaling FFAG will cross many integer and half-integer betatron resonances during the 10-20 turns acceleration. There is the expectation that if driving terms are small enough and crossing is fast enough, then there is insufficient time for the betatron amplitudes to grow. The conventional theory of resonance crossing[2, 3, 4] is applied to slow acceleration, over 100s or 1000s of turns. This paper examines whether the rapid parameter changes encountered in the multi-GeV FFAGs, or few-MeV electron model, are compatible with simple theory.

INTRODUCTION

This paper is in two parts. First, an exact treatment[5] of the simple problem of a time-varying oscillator driven by a sinusoidal forcing that instantaneously matches its natural frequency. Second, an approximate treatment[6] of a complicated ring-type accelerator with alternating-gradient focusing, and magnet field errors. The former case sheds light on the latter; in particular for the regime when the approximations break down - namely large slew rate and/or the vicinity of zero natural frequency. In both cases we find changes to the invariant of motion.

Transfer Matrices, etc

Suppose that $F(t)$ and $G(t)$ are known independent solutions of a homogeneous second order differential equation, $\mathcal{D}x = 0$. The general solution is given $\mathbf{x}(t) = \mathbf{T}(t, t_0)\mathbf{x}(t_0)$ where the matrix and vectors are

$$\mathbf{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix},\ \mathbf{M} = \begin{bmatrix} F & G \\ \dot{F} & \dot{G} \end{bmatrix},\ \mathbf{T} = \begin{bmatrix} C & S \\ \dot{C} & \dot{S} \end{bmatrix}. \quad (1)$$

The transfer matrix $\mathbf{T} = \mathbf{M}(t)\mathbf{M}^{-1}(t_0)$ and $\mathbf{M}^{-1}$ is the matrix inverse of $\mathbf{M}$. The transformation is area preserving when the Wronskian $F\dot{G} - G\dot{F}$ is unity.

Invariant of Motion

When $\mathcal{D} = d^2/dt^2 + p^2(t)$, the trajectory is written in quasi-harmonic form:

$$x = \sqrt{2J\beta(t)}\cos[\psi(t)] \quad (2)$$

$$\dot{x} = -\sqrt{2J/\beta}[\sin\psi + \alpha\cos\psi]\ ; \quad (3)$$

$$\alpha = -\dot{\beta}/2\ ,\ \beta\gamma = 1+\alpha^2\ ,\ \dot{\psi} = 1/\beta\ ,\ \dot{\alpha}+\gamma-p^2\beta = 0\ . \quad (4)$$

The single-particle invariant defines an ellipse of area $\pi\varepsilon$:

$$2J = \gamma x^2 + 2\alpha x\dot{x} + \beta(\dot{x})^2 = \varepsilon\ . \quad (5)$$

Transforming the initial vector $\mathbf{x}(t_0)$ according to $\mathbf{x}(t) = \mathbf{T}\mathbf{x}(t_0)$, and transforming α, β, γ according to (4), both to the location t, and forming the quantity (5) we obtain the anticipated invariant $2J =$ emittance. We may ask what is the result of peturbing the ellipse by a displacement of the centroid. Let $\alpha_0, \beta_0, \gamma_0$ be initial values at $t = t_0$. Consider the vector: $\mathbf{x}(t_0)+(A, B)$. Performing the same operations we obtain $2J = \varepsilon + \Delta J$ where:

$$\Delta J = \gamma_0 A^2 + 2\alpha_0 AB + \beta_0 B^2\ . \quad (6)$$

Driven oscillator

The next step is to consider an inhomogeneous equation $\mathcal{D}x = f(t)$ where the forcing, f, is applied starting at τ. The general solution may be derived *exactly*: $\mathbf{x}(t) = \mathbf{T}[\mathbf{x}(t_0) + (A, B)]$. The change of the invariant is given by (6). Here:

$$A = -\int_\tau^t S(s)f(s)ds\ ,\ B = +\int_\tau^t C(s)f(s)ds\ . \quad (7)$$

LINEAR OSCILLATOR

We consider the driven linear oscillator:

$$\ddot{x} + (a+bt)^2 x = c^2\cos(at)\ . \quad (8)$$

We introduce the dimensionless coordinate $s = at$, in terms of which the O.D.E. becomes

$$x'' + (1+s/k)^2 = (c/a)^2\cos(s)\Theta(s-\sigma)\ . \quad (9)$$

The dimensionless parameter $k \equiv a^2/b$ characterises the slew rate. $\Theta = 1$ for $s \geq \sigma$, else zero.

The transfer matrix $\mathbf{T}(s, 0) = \tilde{\mathbf{T}}(s+k, k)$ where $\tilde{\mathbf{T}}$ is the transfer matrix for the simpler O.D.E. $x'' + (s/k)^2 = 0$ that results when the origin is shifted to $s = -k$. We use the subscripts $\pm$ to denote $\pm t \geq 0$ solutions. we form the matrices $\mathbf{M}_\pm$ and their inverses from $F_\pm, G_\pm$; and from these $\tilde{\mathbf{T}}(t, k)_\pm = \mathbf{M}_\pm(t)\mathbf{M}_+(k)$; and hence $\mathbf{T}_\pm(s, 0) = \tilde{\mathbf{T}}_\pm(s+k, k)$. The free oscillations are $\mathbf{x}(s \geq -k) = \mathbf{T}_+(s, 0)\mathbf{x}(s{=}0)$ and $\mathbf{x}(s{\leq}-k) = \mathbf{T}_-(s, 0)\mathbf{x}(s{=}0)$.

It is natural to take the beam ellipse to be upright at $s = 0$; this implies $\alpha_0 = 0$ and $\beta_0 = \gamma_0 = 1$. The change in the invariant is simply $\Delta J = (c/a)^2[A^2 + B^2]$ where

$$A = -\int_\sigma^s \cos t\, S_\pm dt\ , \qquad B = +\int_\sigma^s \cos t\, C_\pm dt\ , \quad (10)$$

and $C_\pm, S_\pm$ are the elements of $\tilde{\mathbf{T}}_\pm(t+k, k)$ taken as appropriate to whether $t < -k$. Substituting $t \to t\sqrt{k}$, the change of the invariant becomes $(c/a)^4 k[\mathcal{A}^2 + \mathcal{B}^2]$ where $\mathcal{A}, \mathcal{B}$ are weakly varying functions of k.

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WKBJ solution The second approximation:

$$F_\pm = \cos(t^2/2k)\sqrt{\pm k/t}\,,\ G_\pm = \pm\sin(t^2/2k)\sqrt{\pm k/t}\,, \tag{11}$$

valid for $\pm t > 0$. The solution is subject to the condition $k \gg 1$. In the limit of $k \gg 1$, $[\mathcal{A},\mathcal{B}] \rightarrow (\sqrt{\pi}/2)[\mathcal{S},\mathcal{C}]$, Fresnel integrals of argument $s/\sqrt{\pi}$.

Exact Solution Valid for $\pm t \geq 0$. Let $\tau \equiv t^2/(2k)$.

$$F_\pm = \frac{\sqrt{\pi}}{2^{3/4}}\sqrt{\pm t}\,J_{-1/4}(\tau)\,,\ G_\pm = \pm\frac{\sqrt{\pi}}{2^{3/4}}\sqrt{\pm t}\,J_{+1/4}(\tau)\,, \tag{12}$$

Here $J_{\pm 1/4}$ are Bessel functions of fractional order.

Single-sided integration

Assume $a, b, k > 0$. All integrals may be computed numerically in the forward direction from $\sigma \geq 0$ to $s > 0$. The integrals converge, quickly, to aymptotic values with residual oscillations. Figures 1,2,3 show how the integrals vary as the upper limit of integration s increases. The Fresnel integrals estimate the asymptotic value of $\mathcal{A}^2 + \mathcal{B}^2$ to be $\pi/8 \approx 0.3927$, independent of k Integration range $s = [0, 4\sqrt{\pi k}]$ in the reduced variable; and $t = s/a$ and $k = a^2/b$. Hence the time range, $t = [0, 4\sqrt{\pi/b}]$, for asymptotic values to establish gets progressively longer as slew rate b is reduced. Throughout the plots $k = 20$, and $\mathcal{A}$ and $\mathcal{B}$ are shown red and blue, respectively.

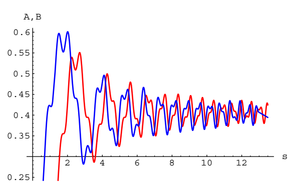

Figure 1: Exact $\mathcal{A}, \mathcal{B}$

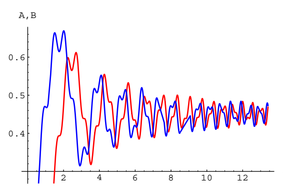

Figure 2: WKBJ $\mathcal{A}, \mathcal{B}$

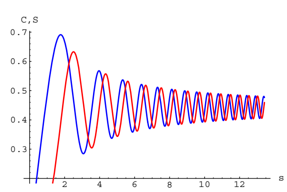

Figure 3: Fresnel $\mathcal{C}, \mathcal{S}$

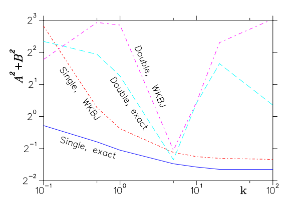

Figure 4: P.I.s versus k

Figure 4 compares the limits of the particular integrals (P.I.s) for single- and double-sided integration for the Bessel and WKBJ kernels as a function of k.

Double-sided integration

When the lower limit σ is extended backward in time, toward $\sigma \rightarrow -k$ and beyond, the integrals vary wildly. The effect is shown in figures 5,6. If the range of integration extends from $t \ll -a/b$ to $t > 4\sqrt{\pi/b}$, the P.I. computed with the WKBJ kernel bears little relationship, except order of magnitude, to the value found with the Bessel kernel. This is because $s = -k$ is territory where the WKBJ solutions become undefined and possibly infinite. Figures 7,8 show $C_\pm, S_\pm$ across $s = -k$. The combination of a singularity and oscillatory terms is very challenging to any algorithm for numerical integration, and that is the source of the high frequency ripple in Fig. 6.

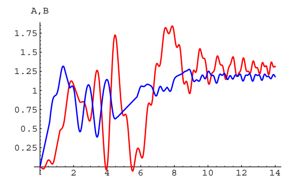

Figure 5: Exact $\mathcal{A}, \mathcal{B}$

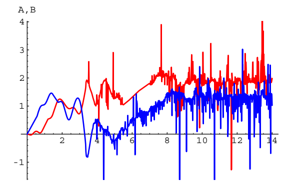

Figure 6: WKBJ $\mathcal{A}, \mathcal{B}$

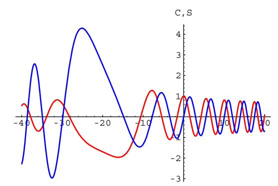

Figure 7: Exact $C_\pm, S_\pm$

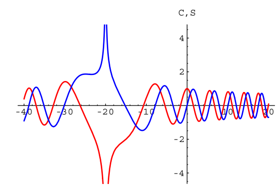

Figure 8: WKBJ $C_\pm, S_\pm$

BETATRON RESONANCE CROSSING

We wish to estimate the change in the transverse invariant. Let $\partial_x^n \equiv \partial^n/\partial x^n$. The ring hamiltonian[7] is $H =$

$$H_0 + \frac{1}{B_0\rho}\sum_{n=1}^{\infty}\frac{1}{n!}\partial_x^{(n-1)}B_z(s)x^n\,,\ H_0 = \frac{p_x^2}{2} + k\frac{x^2}{2}\,, \tag{13}$$

H_0 gives the unperturbed motion. $B_0\rho k(s) = \partial B_z/\partial x$. The terms in the summation over n are small perturbations and will be considered singly. We make canonical transformations to new coordinates $\psi_1 = \psi - \delta\psi$ and J.

$$\delta\psi = [\textstyle\int_0^s du/\beta(u) - (Q/R)s]\,,\ \psi_1' = Q/R\,. \tag{14}$$

The ring tune $Q = \int_0^{2\pi R} ds/(\beta 2\pi)$ where R is the average machine radius and $2\pi R \equiv C$ is its circumference. Let us suppose the lattice tune Q varies linearly: $Q(s) = Q_0 + Q's$ in which case $\psi_1 = [Q_0 s + Q's^2/2]/R$. This phase variation is substituted in the free-oscillation.

Perturbation

We now add the perturbation, writing this in terms of (ψ_1, J). The change of the invariant is essentially the particular integral for the system when it is driven.

$$\frac{1}{J_1^{n/2}}\frac{dJ_1}{ds} = \frac{2^{n/2}}{(n-1)!}\frac{\beta(s)^{n/2}}{B_0\rho}\left[\partial_x^{(n-1)}B_z(s)\right]\cos^{(n-1)}\psi\sin\psi\,. \tag{15}$$

J_1 may be obtained by an integration over the circumferential coordinate s starting with the initial values at location $s = \sigma$. The only completely legitimate procedure is to perform the integration directly. However, it is customary to make the assumption of a resonance condition, and look for dominant contibutions. The lattice terms β, B_z are assumed to be locally periodic functions on the range L, and are expanded as a Fourier series over m. Most terms in

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the integrand are rapidly varying, but we single out those having function argument

$$i2\pi s[(m/L) - n(Q_0/C)] \ - in(Q'/R)(s^2/2) \ , \quad (16)$$

as slowly varying. We identify the resonance condition $m = nQ_0 \times (L/C)$ that must be satisfied by integers n, m. For brevity, let $\eta \equiv (1 - n/2)$ and $\kappa s^2 \equiv nQ's^2/(2R)$. The change of invariant is:

$$\int_{J_1(\sigma)}^{J_1(\sigma+L)} \frac{dJ}{J^{n/2}} = \frac{1}{\eta}[J_1^\eta(L+\sigma) - J_1^\eta(\sigma)] \ . \quad (17)$$

Let the Fourier component $2^{n/2}(n-1)!\,\mathcal{A}_{n,m}(L) \equiv$

$$\frac{1}{B_0\rho}\int_\sigma^{\sigma+L} \beta(s)^{n/2}[\partial_x^{(n-1)}B_z(s)]e^{[in\delta\psi(s)+im2\pi s/L]}ds \ . \quad (18)$$

Ring versus beam line

There are two cases to consider: the linac or beamline in which β, B_z never repeat themselves; and the ring in which $\beta, B_z, \delta\psi$ have minimal periodicity C. In the linac case, the resonant m slides in value as L is extended, yielding:

$$\int_{J(\sigma)}^{J(\sigma+L)} \frac{dJ}{J^{n/2}} = \int_\sigma^{\sigma+L} \Re[-i\mathcal{A}_{n,m(L)}(L)\exp(i\kappa s^2)]ds \ . \quad (19)$$

In the ring case we consider $m = m_0$ to be fixed, and consider the growth on consecutive turns:

$$\int_{J(\sigma)}^{J(\sigma+qC)} \frac{dJ}{J^{n/2}} = \int_\sigma^{\sigma+qC} \Re[-i\mathcal{A}_{n,m_0}(C)\exp(i\kappa s^2)]ds \ , \quad (20)$$

with integer q the turn index.

Notice that the tune is equal to Q_0 at $s = 0$, and the lower limit of integration being $\sigma \leq 0$ allows us to consider a crossing of the resonance. The integrals appearing in (19,20) may be expressed in terms of Fresnel integrals.

Problems

The particular integral for driven motion was constructed from free oscillations. The problem is that (3) are *not* the true oscillations when the betatron tune varies. This defect enabled the P.I. to be approximated by Fresnel integrals, and will founder under the same conditions that invalidate the WKBJ approximation, namely the analogues of $s \approx -k$ and/or $k < 1$.

Conditions on resonance crossing Let N_c, N_t be the number of cells and turns, respectively. Let l_0 be the cell length, then $N_c l_0 = 2\pi R$. Let ν_c be the cell tune and $\Delta\nu = (\nu_c^{\rm ext} - \nu_c^{\rm inj})$ be the difference of cell tunes between injection and extraction. The basic lattice tune is $Q_0 = N_c \times \nu_c$ and the slew rate is $Q' \equiv dQ/ds = \Delta\nu/(N_t l_0)$. $Q' < 0$, $Q = 0$ after beam leaves the ring. Rôles of σ and $\sigma + L$ are reversed. The analogue of a is the rate of phase advance ψ' and hence the WKBJ approximation is valid for

$$(Q/R)^2 \gg |Q'/R| \ , \ \text{ or } \ 2\pi N_c N_t |\nu_c^2/\Delta\nu| \gg 1 \ . \quad (21)$$

The WKBJ approximation is valid provided the upper limit of integration σ satisfies

$$\sigma \ \ll \ Q_0/|Q'| \ = \ -N_c N_t l_0 |\nu_c/\Delta\nu| \ . \quad (22)$$

A further issue is how long must the integration be continued before the integral is replaced by its asymptotic value. The limiting values are obtained when the range of integration $(s = L)$ obeys

$$L \geq 4\sqrt{\frac{\pi}{2\kappa}} = 4\sqrt{\frac{\pi R}{nQ'}} = 4l_0\sqrt{\frac{N_c N_t}{2n\Delta\nu}} \ = \ C\sqrt{\frac{8N_t}{N_c n\Delta\nu}} \ . \quad (23)$$

Rapid acceleration

The linear-field FFAG accelerator has natural chromaticity, and a range of central momenta spanning $\delta p/p = \pm 50$ %; this leads to a large variation of betatron. When used for muons, which decay rapidly, the FFAG must accelerate the beam in a comparably small time span, and so acceleration is rapid: some 10-20 turns in a machine with roughly 100 cells is envisioned. During this time many resonances will be crossed. It is natural to wonder whether the fomulae for change of invariant (or emittance) may be used in such a regime of rapid resonance crossing. The formulae (21-23) allow us to explore this question.

Slew rate For the FFAG at injection, the tune range and cell tune are comparable ($\Delta\nu \simeq -\nu_c^{\rm inj}$) and so the number of cell turns must satisfy the inequality $(2\pi\nu_c)N_c \times N_t \gg 1$. For the FFAG at extraction, the cell tune is a small fraction, f, of the tune range and so the number of cell turns must satisfy $2\pi|\Delta\nu|f^2 N_c \times N_t \gg 1$ where ($\nu_c^{\rm ext} = -f\Delta\nu$). Given that $\nu_c^{\rm inj} \approx 0.4$, $\nu_c^{\rm ext} \approx 0.1$, $N_c \gg 1$ and $N_t \geq 1$, both conditions are satisfied in practical cases.

Range of integration Given that $N_t \geq 10$, and $\Delta\nu \simeq -\nu_c$, the implication of (22) is that we can imagine the end of crossing to occur no later than one turn after being precisely on resonance. Of course, if the particular integral is based on the Bessel function kernel (12), then no such limitation exists.

Equation (23) gives the minimum integration range for the asymptotic value to be established. For values $N_c = 100$, $N_t = 10$, $\Delta\nu = 0.3$ and $n = 1, 2, 3$, the range is roughly C. For those resonances involving only a few cells, this is not satisfied except for rather large n (high-order multipoles). For resonances which we believe to involve only a few cells, we must evaluate $\mathcal{C}(L\sqrt{2\kappa/\pi})$, etc.

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