FREE ELECTRON LASERS WITH SLOWLY VARYING BEAM AND UNDULATOR PARAMETERS*

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Abstract

The performance of a free electron lasers (FEL) is affected when the electron beam energy varies alone the undulator as would be caused by vacuum pipe wakefields and/or when the undulator strength parameter is tapered in the small signal regime until FEL saturation. In this paper, we present a self-consistent theory of FELs with slowly-varying beam and undulator parameters. A general method is developed to apply the WKB approximation to the beam-radiation system by employing the adjoint eigenvector that is orthogonal to the eigenfunctions of the coupled Maxwell-Vlasov equations. This method may be useful for other slowly varying processes in beam dynamics.

INTRODUCTION

High-gain free electron lasers (FEL) are being developed as extremely bright x-ray sources of a next-generation radiation facility. An x-ray FEL based on self-amplified spontaneous emission (SASE) typically requires an electron beam with a few kilo-Ampere peak current and a small-gap undulator system of a hundred meter in length. The collective interaction of a high-current short electron bunch with the undulator vacuum chamber may significantly change the beam energy inside the undulator and degrade the FEL performance, as highlighted by the recent analysis of the ac resistive wall wakefield [1] for the linac coherent light source (LCLS) [2]. The wakefield-induced energy change may be compensated partially by tapering the undulator parameter. In order to evaluate the wakefield effects and to optimize the undulator taper, we develop a self-consistent theory of FELs with slowly varying beam and undulator parameters [3]. In this paper, we show how the WKB method well-known in quantum mechanics can be formulated to solve the coupled Maxwell-Vlasov equations for the FEL system. This approach may be useful for analyses of other slowly varying processes in beam dynamics.

FEL EQUATIONS WITH VARIABLE PARAMETERS

The wakefield generates an energy variation along the undulator distance as well as along the bunch position. Since the typical bunch length for an x-ray FEL greatly exceeds the radiation slippage length over the entire undulator, the energy variation within an FEL slippage length (known as an FEL slice) is usually negligible for the wakefield that do not vary rapidly inside the bunch. Thus, the main effect of the undulator wakefield in an FEL slice is due to the energy change along the undulator distance and may be considered to be equivalent to that caused by tapering the undulator strength parameter.

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Let us consider a planar undulator with a period $\lambda_u = 2\pi/k_u$ and an undulator strength parameter $K(z)$ that may vary along the undulator distance z. We also assume $\gamma_c(z)mc^2$ is the average electron energy in the absence of the FEL interaction, which may vary along the undulator due to wakefields and emission of spontaneous radiation. The initial resonant wavelength of the FEL is

$$\lambda_0 = \frac{2\pi}{k_0} = \frac{2\pi c}{\omega_0} = \frac{\lambda_u}{2\gamma_c(0)^2}\left[1+\frac{K(0)^2}{2}\right]. \quad (1)$$

We define the resonant energy (in units of mc^2) as

$$\gamma_r(z) = \sqrt{\frac{\lambda_u}{2\lambda_0}\left[1+\frac{K(z)^2}{2}\right]}, \quad (2)$$

from which we obtain $\gamma_r(0) = \gamma_c(0) \equiv \gamma_0$. This is the energy of the electron at the undulator location z that radiates at the initial wavelength λ_0.

In this paper, we consider a one-dimensional (1-D) FEL system and ignore any transverse effect. The longitudinal motion of the electron with a wiggle-averaged position ct^* can be described by a ponderomotive phase variable $\theta(z) = (k_0 + k_u)z - k_0 ct^*$ and a normalized energy variable $\eta(z) = [\gamma(z) - \gamma_c(z)]/\gamma_0$. Taking into account that $cdt^*/dz = 1 + (1 + K(z)^2/2)/[2\gamma(z)^2]$ and that changes in K and γ_c over the entire undulator distance are typically very small compared to $K(0) \equiv K_0$ and γ_0, the FEL pendulum equations can be written as

$$\frac{d\theta}{dz} = 2k_u\frac{\gamma(z)-\gamma_r(z)}{\gamma_0} = 2k_u(\eta+\delta), \quad (3)$$

$$\frac{d\eta}{dz} = \frac{eK_0[\mathrm{JJ}]}{4\gamma_0^2 mc^2}\int d\nu E_\nu(z)e^{i\nu\theta - i\Delta\nu k_u z} + \text{c. c.}, \quad (4)$$

where the fractional energy change with respect to the resonant energy in the absence of the FEL interaction is

$$\delta(z) = \frac{\gamma_c(z)-\gamma_r(z)}{\gamma_0} \quad \text{with} \quad \delta(0) = 0. \quad (5)$$

Here $E_\nu(z)$ is the (complex) electric field amplitude at the frequency $\omega = \nu\omega_0$ near ω_0, $\Delta\nu = \nu - 1$, $|\Delta\nu| \ll 1$, and the Bessel function factor [JJ]=$J_0(\xi) - J_1(\xi)$ with $\xi = K_0^2/(4+2K_0^2)$.

In the small signal regime before saturation, the electron distribution function can be decomposed into two parts: a coarse-averaged electron distribution function $V(\eta)$ (for a uniform bunch current) and a small perturbation containing the initial shot noise fluctuation and the

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FEL interaction $\delta F(\theta, \eta; z)$. Incorporating the pendulum Eqs. (3) and (4), the linearized Vlasov equation for the Fourier component of the distribution function $F_\nu(\eta; z) = \int \delta F(\theta, \eta; z) \exp(-i\nu\theta) d\theta/(2\pi)$ is

$$\frac{dF_\nu}{dz} + i\nu 2k_u (\eta + \delta) F_\nu + \frac{eK_0[\mathrm{JJ}]E_\nu(z)}{4\gamma_0^2 mc^2} e^{-i\Delta\nu k_u z} \frac{dV}{d\eta} = 0 \,. \tag{6}$$

The Maxwell equation for the electric field is then

$$\frac{dE_\nu}{dz} = -\frac{ek_0 K_0[\mathrm{JJ}]}{2\epsilon_0 \gamma_0} e^{i\Delta\nu k_u z} \int_{-\infty}^{\infty} d\eta F_\nu(\eta; z) \tag{7}$$

with ϵ_0 being the vacuum permittivity.

Let us introduce the following dimensionless variables:

$$\begin{aligned} \bar{z} &= 2\rho k_u z \,, \quad \bar{\eta} = \frac{\eta}{\rho} = \frac{\gamma(z) - \gamma_c(z)}{\gamma_0 \rho} \,, \\ \bar{\nu} &= \frac{\Delta\nu}{2\rho} \,, \quad \bar{\delta} = \frac{\delta}{\rho} = \frac{\gamma_c(z) - \gamma_r(z)}{\gamma_0 \rho} \,, \\ a_\nu &= -\frac{eK[\mathrm{JJ}]e^{-i\Delta\nu k_u z} E_\nu}{4\gamma_0^2 mc^2 k_u \rho} \,, \quad f_\nu = \frac{2k_u \rho^2}{k_0} F_\nu \,. \end{aligned} \tag{8}$$

Here ρ is the FEL scaling parameter given by [4]:

$$\rho = \left[\frac{1}{8\pi} \frac{I_e}{I_A} \left(\frac{K_0[\mathrm{JJ}]}{1 + K_0^2/2} \right)^2 \frac{\gamma_0 \lambda_0^2}{\Sigma_A} \right]^{1/3} \,, \tag{9}$$

where I_e is the electron peak current, $I_A = 4\pi\epsilon_0 mc^3/e \approx$ 17 kA is the Alfvén current, Σ_A is the area of the electron beam transverse cross section. The Maxwell-Vlasov equations (7) and (6) in the matrix form are

$$\frac{d}{d\bar{z}} \begin{pmatrix} a_\nu \\ f_\nu \end{pmatrix} = i\mathrm{M} \begin{pmatrix} a_\nu(\bar{z}) \\ f_\nu(\bar{\eta}; \bar{z}) \end{pmatrix} \,, \tag{10}$$

where

$$\mathrm{M} = \begin{pmatrix} -\bar{\nu} & -i\int_{-\infty}^{\infty} d\bar{\eta} \\ -i\frac{dV}{d\bar{\eta}} & -\left[\bar{\eta} + \bar{\delta}(\bar{z})\right] \end{pmatrix} \,. \tag{11}$$

We define $\int_{-\infty}^{\infty} d\bar{\eta}$ as the integration operator that operates on a function of $\bar{\eta}$.

WKB SOLUTION

Since the main effect of the energy variation is to move electrons off-resonance, $\delta(z)$ can be regarded as a slowly varying function of z when the relative energy change per field gain length ($\sim \lambda_u/(4\pi\rho)$) is much less than the relative gain bandwidth that is typically a few ρ, i.e.,

$$\left| \frac{\lambda_u}{4\pi\rho} \frac{d\delta}{dz} \right| \ll \text{a few } \rho \,, \quad \text{or} \quad \left| \frac{d\bar{\delta}}{d\bar{z}} \right| < 1 \,. \tag{12}$$

This condition allows us to use the WKB approximation to solve Eq. (10) and is satisfied if the accumulated energy change over the saturation distance (typically about 10 field gain length) is less than 10ρ.

WKB approximation using the matrix formalism

We first illustrate the WKB approximation using the matrix formalism. Consider a second-order differential equation for a function $\phi(x)$:

$$\frac{d^2\phi}{dx^2} + k^2(x)\phi(x) = 0 \,, \tag{13}$$

where the parameter $k(x)$ is assumed to be slowly varying in x. Equation (13) can be regarded as the one-dimensional, time-independent Schrödinger equation for the wavefunction $\phi(x)$. Let us define $d\phi/dx \equiv \varphi$ and convert Eq. (13) to a couple of first-order differential equations. Using the matrix notation, we have

$$\frac{d}{dx} \begin{pmatrix} \phi \\ \varphi \end{pmatrix} = \mathrm{L} \begin{pmatrix} \phi \\ \varphi \end{pmatrix} \,, \quad \mathrm{L} = \begin{pmatrix} 0 & 1 \\ -k^2(x) & 0 \end{pmatrix} \,. \tag{14}$$

Since $k(x)$ is slowly varying, we expect the solution to closely approximate the free-particle state, i.e.,

$$\begin{pmatrix} \phi \\ \varphi \end{pmatrix} \approx \Psi_0(x) e^{iS_0(x)} \,, \quad \text{with} \quad \frac{dS_0}{dx} = \pm k(x) \,. \tag{15}$$

The eigenvector given by $\pm ik\Psi_0 = \mathrm{L}\Psi_0$ is

$$\Psi_0^+(x) = \begin{pmatrix} 1 \\ ik(x) \end{pmatrix} \,, \quad \Psi_0^-(x) = \begin{pmatrix} 1 \\ -ik(x) \end{pmatrix} \,. \tag{16}$$

We define Φ_0 as the adjoint eigenvector that satisfies $\pm ik\Phi_0 = \Phi_0 \mathrm{L}$ and find

$$\Phi_0^+(x) = \left(1, \frac{1}{ik(x)}\right) \,, \quad \Phi_0^-(x) = \left(1, -\frac{1}{ik(x)}\right) \,. \tag{17}$$

The adjoint eigenvector $\Phi_0^\pm$ is orthogonal to the eigenvector $\Psi_0^\pm$ since the scalar products $\Phi_0^\pm \Psi_0^\mp = 0$ and $\Phi_0^\pm \Psi_0^\pm = 2$.

To take into account the slow variation of $\Psi_0(x)$ in Eq. (15), we introduce first-order corrections as

$$\begin{pmatrix} \phi \\ \varphi \end{pmatrix} \approx \left[\Psi_0(x) + \Psi_1(x)\right] e^{i[S_0(x) + S_1(x)]} \,, \tag{18}$$

where Ψ_1 and dS_1/dx are considered small, but not S_1. Inserting this into Eq. (14) and ignoring higher-order terms $d\Psi_1/dx$ and $\Psi_1 dS_1/dx$, we have

$$\frac{d\Psi_0}{dx} + i\frac{dS_1}{dx}\Psi_0 = \left(-i\frac{dS_0}{dx} + \mathrm{L}\right)\Psi_1 \,. \tag{19}$$

Applying the adjoint eigenvector Φ_0 to Eq. (19), the scalar product of the right hand side vanishes because $\Phi_0(idS_0/dx) = \Phi_0 L$, and the scalar product of the left hand side becomes

$$\Phi_0 \left[\frac{d\Psi_0}{dx} + i\frac{dS_1}{dx}\Psi_0\right] = \frac{1}{k}\frac{dk}{dx} + 2i\frac{dS_1}{dx} = 0 \,, \tag{20}$$

from which we obtain $S_1 = i \ln \sqrt{k(x)}$. Inserting S_0 and S_1 into Eq. (18) and neglecting Ψ_1 in comparison with Ψ_0, we obtain the standard WKB solution

$$\phi(x) \approx \frac{C_1}{\sqrt{k(x)}} \exp\left[i\int k(x)dx\right] + \frac{C_2}{\sqrt{k(x)}} \exp\left[-i\int k(x)dx\right], \qquad (21)$$

where C_1 and C_2 are given by initial/boundary conditions.

FEL growth rates

Following the above discussion, we seek a zeroth-order solution of Eq. (10) in the form

$$e^{-i\int_0^{\bar z}\mu_0(\tau)d\tau}\Psi_0 \equiv e^{-i\int_0^{\bar z}\mu_0(\tau)d\tau}\begin{pmatrix} A_0 \\ \mathcal{F}_0(\bar\eta;\bar z)\end{pmatrix}. \qquad (22)$$

In the 1-D case, A_0 is simply a constant given by the initial conditions. Treating $d\mathcal{F}_0/d\bar z$ as a first-order term, the zeroth-order eigenvalue equation is

$$\begin{pmatrix} (\mu_0-\bar\nu) & -i\int_{-\infty}^{\infty} d\bar\eta \\ -i\frac{dV}{d\bar\eta} & \left[\mu_0-\left(\bar\eta+\bar\delta(\bar z)\right)\right]\end{pmatrix}\begin{pmatrix} A_0 \\ \mathcal{F}_0(\bar\eta)\end{pmatrix} = 0\,. \qquad (23)$$

The eigenvalue is determined by solving the second row for

$$\mathcal{F}_0(\bar\eta;\bar z) = \frac{iA_0}{\mu_0-\left[\bar\eta+\bar\delta(\bar z)\right]}\frac{dV}{d\bar\eta} \qquad (24)$$

and inserting it into the first row. The dispersion relation is

$$\mu_0-\bar\nu = \int_{-\infty}^{\infty}\frac{d\bar\eta}{\left[\bar\eta+\bar\delta(\bar z)-\mu_0\right]}\frac{dV}{d\bar\eta}\,, \qquad (25)$$

or

$$\hat\mu-\hat\nu = \int_{-\infty}^{\infty}\frac{d\bar\eta}{(\bar\eta-\hat\mu)}\frac{dV}{d\bar\eta}\,, \qquad (26)$$

$$\text{with}\quad \hat\mu(\bar z)\equiv\mu_0(\bar z)-\bar\delta(\bar z)\,,\quad \hat\nu(\bar z)\equiv\bar\nu-\bar\delta(\bar z)\,. \qquad (27)$$

This is the same FEL dispersion relation as in the constant-parameter case [5]. For a variable-parameter FEL, the instantaneous frequency detune $\hat\nu(\bar z) = \bar\nu - \bar\delta(\bar z)$ is $\bar z$-dependent due to changes in the beam energy and the undulator parameter. As a result, the local growth rate Im(μ_0)=Im($\hat\mu$) is also a function of $\bar z$. The corresponding eigenvector is

$$\Psi_0(\bar z) = \begin{pmatrix} A_0 \\ \mathcal{F}_0(\bar\eta;\bar z)\end{pmatrix} \propto \begin{pmatrix} 1 \\ \frac{i}{\mu_0-\left[\bar\eta+\bar\delta(\bar z)\right]}\frac{dV}{d\bar\eta}\end{pmatrix}. \qquad (28)$$

To take into account the z-dependence of $\mathcal{F}_0$, we must include the first-order corrections as

$$\begin{pmatrix} a_\nu \\ f_\nu\end{pmatrix} \approx e^{-i\int_0^{\bar z}[\mu_0(\tau)+\mu_1(\tau)]d\tau}\left[\Psi_0(\bar z)+\Psi_1(\bar z)\right]. \qquad (29)$$

Note that both $\Psi_1 = (A_1, \mathcal{F}_1(\bar\eta))$ and μ_1 are considered small as compared to Ψ_0 and μ_0, respectively, but the accumulated phase change $\int_0^{\bar z}\mu_1(\tau)d\tau$ in the exponent can be of the same order. Inserting Eq. (29) into Eq. (10) yields

$$\left[-i\mu_0(\bar z)-i\mu_1(\bar z)\right](\Psi_0+\Psi_1)+(\Psi_0'+\Psi_1') = i\mathrm{M}(\Psi_0+\Psi_1)\,, \qquad (30)$$

where $(') = d/d\bar z$. Making use of $-i\mu_0\Psi_0 = i\mathrm{M}\Psi_0$ and neglecting the higher-order terms $\mu_1\Psi_1$ and Ψ_1', we have

$$\Psi_0' - i\mu_1\Psi_0 = i(\mu_0+\mathrm{M})\Psi_1\,. \qquad (31)$$

The growth rate correction μ_1 can be found by using an adjoint eigenvector and a properly defined scalar product [3]. In the 1-D case, the adjoint eigenvector is simply

$$\Phi_0 = \left(1, \frac{i}{\mu_0-\left[\bar\eta+\bar\delta(\bar z)\right]}\right). \qquad (32)$$

Defining the 1-D scalar product as

$$\langle\Phi_0|\Psi_0\rangle_{\mathrm{1D}} = \left[1-\int_{-\infty}^{\infty}d\bar\eta\frac{dV/d\bar\eta}{\left[\mu_0-\left(\bar\eta+\bar\delta(\bar z)\right)\right]^2}\right] \equiv B\left(\mu_0-\bar\delta\right)\,, \qquad (33)$$

we apply Φ_0 to both sides of Eq. (31). The resulting scalar product of the right side with Φ_0 is

$$i(\mu_0-\bar\nu)A_1+\int_{-\infty}^{\infty}d\bar\eta\mathcal{F}_1(\bar\eta)+ \int_{-\infty}^{\infty}d\bar\eta\left[\frac{iA_1}{\mu_0-\left[\bar\eta+\bar\delta(\bar z)\right]}\frac{dV}{d\bar\eta}-\mathcal{F}_1(\bar\eta)\right] = 0 \qquad (34)$$

in view of the dispersion relation Eq. (25). Thus, the scalar product of the left side of Eq. (31) with Φ_0 is

$$-i\mu_1B\left(\mu_0-\bar\delta\right)+\int_{-\infty}^{\infty}d\bar\eta\frac{(\mu_0'-\bar\delta')dV/d\bar\eta}{\left[\mu_0-\left(\bar\eta+\bar\delta(\bar z)\right)\right]^3} = 0\,, \qquad (35)$$

Using variables defined in Eq. (27), the correction to the complex growth rate is

$$\mu_1 = -i\frac{\mu_0'-\bar\delta'}{B\left(\mu_0-\bar\delta\right)}\int_{-\infty}^{\infty}d\bar\eta\frac{dV/d\bar\eta}{\left[\mu_0-\left(\bar\eta+\bar\delta(\bar z)\right)\right]^3} = -i\frac{\hat\mu'}{B\left(\hat\mu\right)}\int_{-\infty}^{\infty}d\bar\eta\frac{dV/d\bar\eta}{\left(\hat\mu-\bar\eta\right)^3}\,, \qquad (36)$$

which can be obtained after solving the FEL dispersion relation (i.e., Eq. (25) or (26)).

The FEL growth rates predicted by Eqs. (26) and (36) agree well with 1-D FEL simulations and are used in Ref. [3] to study the LCLS wakefield effects and to optimize the undulator taper. Start-to-end LCLS FEL simulation results are reported elsewhere in these proceedings [6].

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