ANALYTICAL CONSIDERATIONS FOR REDUCING THE EFFECTIVE EMITTANCE WITH VARIABLE DIPOLE FIELD STRENGTHS

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Abstract

The basic optics design scope in lepton rings is to match the sections in either side of the bending magnets in order to minimise the equilibrium emittance. A further important emittance reduction can be achieved by incorporating dipoles for which the deflecting field varies along the electron beam path in the magnet. The figure of merit for such lattices when used in a synchrotron light source is the minimisation of the so-called effective emittance. The effective emittance is computed in the middle of the undulator straight section as the product of the rms size and divergence and therefore includes contributions from the betatron emittance and from the electron energy spread. In this paper, analytical formulas are obtained for the minimum betatron and effective emittance in arbitrary dipole fields and the associated optics functions at the dipole entrance. Examples are given for specific dipole field functions and their properties with respect to the effective emittance minimisation. Finally, the effective emittance is parameterised with respect to standard cell optics properties.

EFFECTIVE EMITTANCE IN ARBITRARY BENDING FIELDS

In a ring without vertical bending, the synchrotron radiation damping is balanced out by the quantum excitation in the horizontal plane only, resulting in an equilibrium betatron emittance

$$\epsilon_x = \frac{C_q\gamma^2}{J_x}\frac{\oint \frac{\mathcal{H}_x(s)}{|\rho_x|^3}ds}{\oint \frac{1}{\rho_x^2}ds} \quad . \tag{1}$$

When dispersion is present in the straight section, the figure of merit for increasing the brilliance at the insertion device (ID) is the effective emittance [1]

$$\epsilon_{x_{eff}}(s_{ID}) = \sqrt{\epsilon_x^2 + \mathcal{H}_x(s_{ID})\epsilon_x\sigma_\delta^2} \ , \tag{2}$$

where $\mathcal{H}_x(s) = \beta_x{\eta'}_x^2 + 2\alpha_x\eta_x\eta'_x + \gamma_x\eta_x^2$ is a function of the optics around the ring, which is invariant outside the bending magnets (neglecting the effect of the IDs). The equilibrium energy spread is defined as $\sigma_\delta^2 = \frac{C_q\gamma^2}{J_s}\frac{\oint \frac{1}{|\rho_x|^3}ds}{\oint \frac{1}{\rho_x^2}ds}$. Assuming that the effect of focusing in the bending magnet is small, the longitudinal damping partition number is $J_s \approx 2J_x$.

In isomagnetic lattices, the betatron emittance is proportional to $\frac{\oint \mathcal{H}_x(s)ds}{|\rho_x|}$ which yields an effective emittance proportional to the third power of the bending angle $\theta^3 \propto N^{-3}$, i.e. it decreases rapidly with the number of dipoles N installed in the ring. An interesting idea [2, 3] would be to vary the bending field along the dipole in order to further reduce the emittance by matching the bending field variation to the variation of the function $\mathcal{H}$.

In order to obtain general formulas for the reduction of the emittance in arbitrary dipole fields, we consider the transport matrix of a generalized dipole magnet with varying bending field, in thin lens approximation and ignoring edge focusing

$$\mathcal{M}_{bend} = \begin{pmatrix} 1 & s & \widetilde{\theta(s)} \\ 0 & 1 & \theta(s) \\ 0 & 0 & 1 \end{pmatrix} \ , \tag{3}$$

where $\theta(s)$ is the bending angle and $\widetilde{\theta(s)}$ its integral along the magnet:

$$\theta(s) = \int_0^s \frac{ds}{\rho(s)} \ , \ \widetilde{\theta(s)} = \int_0^s\int_0^s \frac{d^2s}{\rho(s)} \ . \tag{4}$$

Having as initial values β_0, α_0, γ_0, η_0, and η'_0, at the bending magnet entrance (ID side), the horizontal optics functions evolve as

$$\begin{aligned} &\beta(s) = \beta_0 - 2s\alpha_0 + s^2\gamma_0 \ , \ \ \alpha(s) = \alpha_0 - s\gamma_0 \ , \\ &\eta(s) = \eta_0 + s\eta'_0 + \widetilde{\theta(s)} \ , \ \ \eta'(s) = \eta'_0 + \theta(s) \end{aligned} \tag{5}$$

and $\gamma(s) = \gamma_0$, along the generalised bending magnet. Replacing the optics function evolution in $\mathcal{H}$ and integrating around the ring, the following expression for the betatron equilibrium emittance is obtained

$$\begin{aligned} \epsilon_x = \frac{C_q\gamma^2}{J_x\oint\frac{1}{\rho_x^2}ds}\Big[&\beta_0(A_3 + 2A_2{\eta_0}' + A_1{\eta_0}'^2) \\ &+2\alpha_0(A_5 + A_4{\eta_0}' + \eta_0(A_2 + A_1{\eta_0}')) \\ &+\gamma 0(A_6 + 2A_4\eta_0 + A_1{\eta_0}^2)\Big] \end{aligned} \quad . \tag{6}$$

The betatron emittance depends on six integrals involving the variable bending radius $\rho(s)$:

$$\begin{aligned} &A_1 = \oint \frac{1}{|\rho^3|}ds, \ A_2 = \oint \frac{\theta}{|\rho^3|}ds, \ A_3 = \oint \frac{-\theta^2}{|\rho^3|}ds, \\ &A_4 = \oint \frac{\widetilde{\theta} - s\theta}{|\rho^3|}ds, \ A_5 = \oint \frac{(\widetilde{\theta} - s\theta)^2}{|\rho^3|}ds, \\ &A_6 = -\oint \frac{\theta(\widetilde{\theta} - s\theta)}{|\rho^3|}ds \ . \end{aligned} \tag{7}$$

Using $J_s = 2J_x$ and setting $C = \frac{C_q\gamma^2}{2J_x\oint\frac{1}{\rho_x^2}ds}$, the effective emittance at the ID is

$$\epsilon_{x_{eff}}^2 = \epsilon_x(2\epsilon_x + CA_1\mathcal{H}_x(s_{ID})) \ , \tag{8}$$

i.e., it depends on the initial optics functions, the integrals A_{1-6} and the value of the invariant $\mathcal{H}$ at the ID, for which $\mathcal{H}_x(s_{ID}) = \mathcal{H}_x(s_0)$. Imposing that the partial derivatives of (8) vanish with respect to the initial optics functions in order to get the effective emittance minimum, the following expressions are obtained

$$\eta_0 = \frac{A_4}{A_2}\eta_0' \ , \gamma_0 = \frac{A_2(A_3 + A_2\eta_0')\beta_0}{2A_2A_6 + A_4^2\eta_0'} \ , \qquad \alpha_0 = -\frac{A_2(A_5 + A_4\ \eta_0')\beta_0}{2A_2A_6 + A_4^2\eta_0'} \ , \tag{9}$$

involving the optics functions at the entrance of the bend and the integrals A_{1-6}. Eliminating the other optics functions, it remains a third order polynomial equation of η_0', whose real solution can be replaced back to (9) in order to get the optics functions at the entrance of the bend and the value of the minimum effective emittance (see Appendix).

Minimum betatron emittance: In damping rings of e^+e^- colliders, it is essential to minimise the betatron emittance, through minimum emittance cells [4]. In this case in order to obtain an analytic expression for generalised bendings one has to find the initial optics functions minimising expression (6). Setting $A = (2A_2A_4 - A_1A_5)A_5 - A_2^2A_6 + A_3(A_1A_6 - A_4^2)$, the optics functions giving the minimum betatron emittance $\epsilon_{x;min} = \frac{2C\sqrt{A_1A}}{A_1}$ are [2]:

$$\eta_0 = -\frac{A_4}{A_1} \ , \beta_0 = \frac{-A_4^2 + A_1A_6}{\sqrt{A_1A}} \ , \qquad \eta_0' = -\frac{A_2}{A_1} \ , \alpha_0 = \frac{A_2A_4 - A_1A_5}{\sqrt{A_1A}} \ . \tag{10}$$

Minimum emittance in an achromat: Imposing achromatic conditions $\eta_0 = \eta_0' = 0$ at the dipole entrance, the minimum betatron emittance is equal to the effective one $\epsilon_{x;min} = 2C\sqrt{A_3A_6 - A_5^2}$ and is reached for the simple optics conditions

$$\beta_0 = \frac{A_6}{\sqrt{A_3A_6 - A_5^2}} \ , \alpha_0 = \frac{A_5}{\sqrt{A_3A_6 - A_5^2}} \ . \tag{11}$$

Constant bending radius: For a uniform bending magnet, the following expression for the effective emittance $\epsilon_{x;eff_{min}} = 0.03339 C_q \frac{\gamma^2\theta^3}{J_x}$ is obtained [1], which is a factor 1.55 higher than the minimum achievable betatron emittance for uniform bends $\epsilon_{x_{min}} = \frac{1}{12\sqrt{15}} C_q \frac{\gamma^2\theta^3}{J_x}$.

The question a lattice designer often asks is by how much the emittance grows when the optics functions are detuned from their optimal values [4]. Using Eq. (8), the expression of the minimum emittance and the optimal optics functions for reaching it, a parametric equation can be formed. It is a 4th order polynomial involving the optics functions normalised by their optimal values and parameterised by the ratio of the effective emittance reached with respect to its absolute minimum. By keeping α, η' to the optimal values and letting β, η to vary, curves of constant emittance can be plotted. In the case of the betatron emittance, these curves are ellipses [4]. In the case of the effective emittance, these are distorted ellipses, as shown in

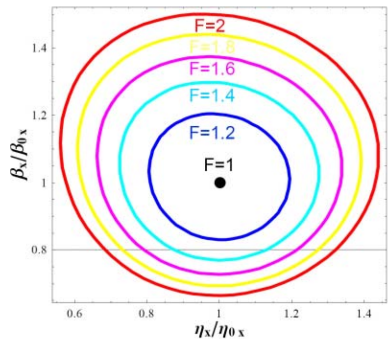

Figure 1: Constant emittance curves when the beta and dispersion functions are detuned from the optimal values.

Fig. 1, in the case of the constant field dipole of the ESRF type. This plot shows that in order to have less than a factor of 1.2 of effective emittance increase, the initial β and η functions have to vary by less than 20% with respect to their optimal values. Similar arguments can be concluded for all optics functions.

NUMERICAL EXAMPLE

Following [3], a bending radius is considered evolving as $\rho_x(s) = (1 + as)^m/b$. Fixing the total bending angle $\theta = \frac{b(1-(1+a\ l_d)^{1-m}}{a(m-1)}$ imposes that $b = \frac{2a(m-1)\pi}{N(1+(1+a\ l_d)^{1-m}}$, where N is the number of dipole in the ring. For the numerical evaluation, the ring layout of the ESRF storage ring is used, with 32 cells (64 dipoles) in a ring circumference of 844.4 m with a beam energy of 6 GeV. We employed the length of the actual ESRF dipole of around 2.33 m and an effective bending radius of 22.89 m, giving an effective dipole field of 0.85 T. In Fig. 2, we plot the minimum effective emittance based on this bending radius profile, versus the degree of the polynomial m and for different values of the field parameter a. The effective emittance drops radically to below 0.5 nm when increasing a and for moderate values of m. For large values of m, it seems to saturate to around 0.6 nm, for all a. The emittance minimum in the case of an achromatic cell is between 1.4 and 2 times larger than the one of the ring with dispersive straight sections (Fig. 2b). For large values of m, it converges towards a ratio of 1.6.

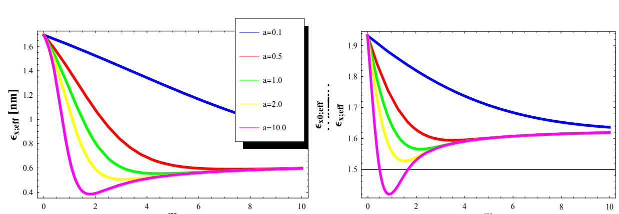

Figure 2: Minimum effective emittance (left) and its ratio with respect to the minimum effective emittance in the achromatic case (right) as a function of the field parameter m for different values of a.

The effective emittance shows a minimum for certain values of the field parameters, which becomes more pronounced for larger values of a. For each value of a, the

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corresponding m can be numerically tracked, where the effective emittance presents a global minimum. In Fig. 3, the global minimum values of the effective emittance versus the field parameters a and m is presented, including their mutual dependence. The global minimum grows for increasing values of a and decreasing values of m. Unfortunately, for this profile model, the corresponding maximum bending fields giving global effective emittance minima, are above 6 T, which can not be produced with normal conducting magnets (Fig. 3d). Indeed, if the field is not constrained, one can reach a zero effective emittance theoretically, as a grows to infinity.

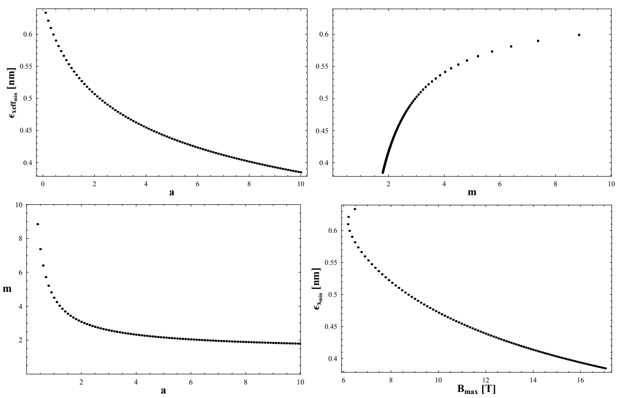

Figure 3: Global minimum of the effective emittance (top) versus the field parameters a (left) and m (right), their mutual dependence (bottom left) and dependence of the global minimum to the maximum bending field (bottom right).

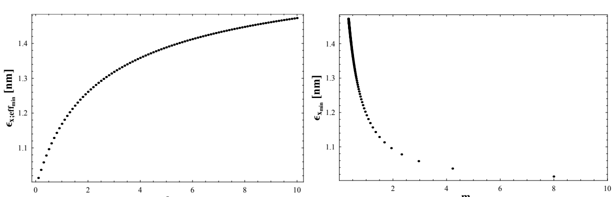

Figure 4: Dependence of the minimum effective emittance in the field parameters a (left) and m (right) in the case of a fixed maximum bending field of 1.8 T.

Therefore, it is interesting to constrain the maximum bending field to normal conducting values. By fixing the maximum dipole field, the values of the parameters a and m are dependent on each other through equation $B = \frac{10}{2.998}\frac{a\sqrt{E^2-E_0^2}(m-1)2\pi}{N(1+(1+a\, l_d)^{1-m}(1+as)^m)}$. In Fig. 4,the minimum effective emittance is plotted versus the two field parameters which are now linked by the previous relation in order to keep the maximum field to 1.8 T. The minimum emittance value drops as expected from 1.69 nm for $m = 0$ (constant field) to below 1.1 nm. The drawback for using this field profile is that, in order to diminish the effective emittance below 1 nm, which is the target value for an ESRF lattice upgrade [5], the order of the bending polynomial has to be raised above $m = 10$ and the field parameter has to drop below $a = 0.1$. In the case of the ESRF, a constant field profile in three steps was preferred, whose length and field values can be optimised numerically to give a minimum effective emittance of 0.77 nm [5].

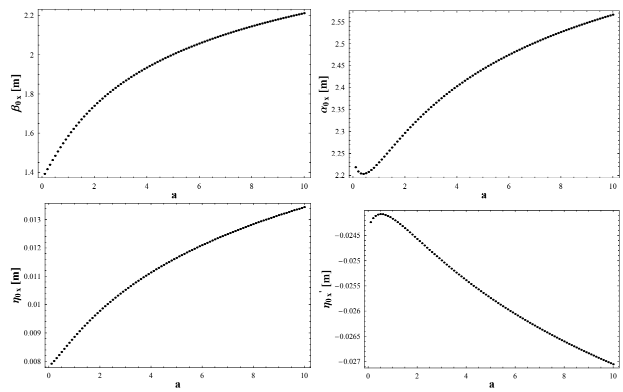

Figure 5: Dependence of horizontal beta, alpha (top), dispersion function and its derivative (bottom) through the bending magnet, on the bending parameter a for a maximum bending field of 1.8 T.

In Fig. 5, we plot the optics functions versus the field parameter a, giving the minimum effective emittances of Fig. 4. The absolute value of all optics function drops with a, i.e. with the effective emittance. Note that the very small values of the dispersion and its derivative indicate that by imposing these optics at the beginning of the bend, a very large phase advance has to be achieved with important implications in the optics design and non-linear dynamics [5].

APPENDIX

Using (9), a third order polynomial depending only on η_0' has to be solved: $3{\eta_0'}^3 + 10T_1{\eta_0'}^2 + T_1^2(6-5T_2)\eta_0' - 4T_1^3T_2 = 0$ with $T_1 = \frac{A_2}{A_1}$, $T_2 = \frac{A_1(A_5^2-A_3A_6)}{A_3A_4^2+A_2(-2A_4A_5+A_2A_6)}$ $T_3 = A_3A_4^2 + A_2(-2A_4A_5 + A_2A_6)$. The optics functions for the minimum effective emittance are:

$$\beta_0 = \frac{9A_1A_6T + A_4^2(46+(T-10)T+45T_2)}{3\sqrt{A_1T_3T(46+T(T-10-9T_2)+45T_2}},$$

$$\alpha_0 = \frac{-A_1(9A_5T + A_4T_1(46+(T-10)T+45T_2)}{3\sqrt{A_1T_3T(46+T(T-10-9T_2)+45T_2}},$$

$$\eta_0 = \frac{A_4(T-10+\frac{46+45T_2}{T})}{9A_1}, \eta_0' = \frac{T_1(T-10+\frac{46+45T_2}{T})}{9},$$

with $T_3 = A_3A_4^2 + A_2(A_2A_6 - 2A_4A_5)$ and $T = (9(-3(1+T_2)(2+3T_2)(126+125T_2))^{1/2} - 190 - 189T_2)^{\frac{1}{3}}$. Then minimum effective emittance is

$$\epsilon_{x_{eff}}^2 = \frac{C^2T_3P_1(T,T_2)P_2(T,T2)}{486A_1T^3(46+T(T-9T_2-10)+45T_2)}$$

with $P_1(T,T_2) = T^4 - 2T^3 - 6T^2(3T_2-2) - 2T(45T_2+46) + (45T_2+46)^2$ and $P_2(T,T_2) = T^4 + 7T^3 - 6T^2(12T_2+13) + 7T(45T_2+46) + (45T_2+46)^2)$.

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