CALCULATION OF REFLECTION MATRIX ELEMENTS OF A GRATING FOR GROWING EVANESCENT WAVES *

Vinit Kumar†, Kwang-Je Kim, ANL, Argonne, IL 60439, USA

Abstract

Reflection matrix elements of a grating play an important role in the study of Smith-Purcell (SP) free-electron lasers (FELs). Especially, the matrix element $\mathcal{R}_{00}$, which couples the incident co-propagating evanescent wave to the reflected co-propagating evanescent wave, is important for evaluation of the gain of an SP-FEL system [1]. We present a calculation of $\mathcal{R}_{00}$ for rectangular grating and study its frequency dependence for a given phase velocity of incident wave. For the numerical calculation, we use the modal expansion method and extend it to include waves having slowly varying amplitude. The singularity of $\mathcal{R}_{00}$ is studied in some detail and we find that it is possible to get a simple formula for the location of the singularity if we choose the eigenmodes of the groove as a basis set as done by Andrews et al. [2].

INTRODUCTION

Devices like Smith-Purcell (SP) free-electron lasers (FELs) and surface wave accelerators are based on the interaction of the electron beam with the co-propagating evanescent wave supported by the grating. In the case of SP-FELs, as shown by Toraldo de Francia [3], the incident evanescent wave from the electron beam gets reflected at the reflection grating to various spectral orders with an amplitude chacterized by the reflection matrix $\underline{\mathcal{R}}$. The matrix element $\mathcal{R}_{mn}$ is defined as the ratio of the amplitude of the m^{th}-order reflected wave to the amplitude of the n^{th}-order incident wave when only n^{th}-order is present in the incident wave. The knowledge of the reflection matrix is crucial to developing an understanding of these devices.

Evaluation of the reflection matrix elements of a grating is an involved problem and has attracted considerable interest for nearly one hundred years [4,5]. In the case of SP radiation, one needs to evaluate the matrix elements for incident $evanescent$ wave. Matrix elements $\mathcal{R}_{m0}$ were first evaluated for reflected propagating waves by Van den berg and the spontaneous SP spectrum was thus calculated [6]. For the case of stimulated emission in SP-FELs, it is more relevant to study the matrix element $\mathcal{R}_{00}$ for the reflected zeroth-order evanescent wave, which co-propagates with the electron beam [1,7]. To the best of our knowledge, the calculation of $\mathcal{R}_{00}$ has not been reported so far. In this paper, we perform a calculation of $\mathcal{R}_{00}$ for a rectangular grating, taking the slow variation in amplitude of the wave into account, and present the results.

* Work supported by U.S. Department of Energy, Office of Basic Energy Sciences, under Contract No. W-31-109-ENG-38.

† vinit@aps.anl.gov

BASIC THEORY

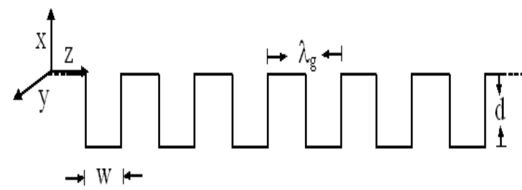

Figure 1: Schematic of a rectangular reflection grating. The top surface of the grating is in the plane x = 0.

Figure 1 shows the schematic of a rectangular metallic reflection grating having period λ_g, groove depth d, and groove width w. Assuming translational invariance in y, the EM field in the region $x > 0$ is composed of the incident and reflected wave having the following Floquet-Bloch expansion for the TM mode:

$$H_y^I = \sum_{n=-\infty}^{+\infty} A_n^I e^{(i\alpha_n z - ip_n x - i\omega t)}, \tag{1}$$

$$H_y^R = \sum_{n=-\infty}^{+\infty} A_n^R e^{(i\alpha_n z + ip_n x - i\omega t)}. \tag{2}$$

Here n is the spectral order, $A_n^R = \sum \mathcal{R}_{nm} A_m^I$, $\alpha_n = k/\beta - nk_g - i\mu$, $k = \omega/c$, $k_g = 2\pi/\lambda_g$, $p_n = i\Gamma_n = \sqrt{k^2 - \alpha_n^2}$, and the sign of the square root is chosen such that $[\mathrm{Re}(p_n) + \mathrm{Im}(p_n)] \geq 0$, which is essentially the outgoing wave condition [5]. Note that we have assumed a slow variation in the amplitude of the type $e^{\mu z}$ for x = 0, where μ is the complex growth rate. The phase velocity of the zeroth order mode is β in units of the speed of light c.

The electromagnetic field inside the groove can be expressed as a superposition of cavity modes and by satisfying the boundary condition at the surface x = 0, we can derive the expression for $\underline{\mathcal{R}}$ and show that [5-7]

$$\underline{\mathcal{R}} = (\underline{I} + \underline{Z})^{-1}(\underline{I} - \underline{Z}), \tag{3}$$

where $\underline{I}$ is the identity matrix and $\underline{Z}$ is the impedence matrix given by

$$Z_{mn} = - \frac{w}{\lambda_g}\frac{1}{\Gamma_m}\sum_{s=0}^{\infty}\frac{Q_s \tan(Q_s w)}{g_s} \times \mathcal{L}_+(p_n; s)\mathcal{L}_-(p_m; s). \tag{4}$$

Here, $Q_s = (k^2 - q_s^2)^{1/2}$, $q_s = \pi s/w$, $g_0 = 1, g_{s\neq 0} =$

$1/2$, and $\mathcal{L}_+$ and $\mathcal{L}_-$ are given by

$$\mathcal{L}_{\pm}(p_m; s) = e^{\pm i\alpha_m b}\frac{1}{2}\left[e^{\mp i\theta_-}\frac{\sin\theta_-}{\theta_-} + e^{\mp i\theta_+}\frac{\sin\theta_+}{\theta_+}\right],$$

where $\theta_{\pm} = (-w\alpha_m \pm \pi s)/2$, and $b = \lambda_g - w$. The above expression for $\underline{\mathcal{R}}$ has the same form as given in Ref. [7], and we have generalized it to take the slow variation of amplitude into account. In the next section, we use this expression to numerically evaluate $\mathcal{R}_{00}$.

NUMERICAL CALCULATIONS

The reflection matrix $\underline{\mathcal{R}}$ is essentially an $\infty \times \infty$ matrix. In order to get a more practical converging solution for $\mathcal{R}_{00}$, we truncate Eq. (3) after the m^{th}-order, i.e., we consider spectral orders from -m to + m and take only the terms from s = 0 to s = m in Eq. (4). We then solve Eq. (3) for a given truncation order m to calculate $\mathcal{R}_{00}$. Next, we look for a converging solution for $\mathcal{R}_{00}$ as m is increased. The calculation is stopped when a prescribed convergence accuracy is achieved, which is 0.1% in our calculation.

We first present the calculation for μ = 0, i.e., ignoring the slow variation in the amplitude. Figure 2 shows the result of this calculation where we have plotted $|\mathcal{R}_{00}|^2$ as a function of free-space wavelength λ $(= 2\pi/k)$. Here, we have used the grating parameters corresponding to the Dartmouth SP-FEL experiment [8], which are β = 0.35, λ_g = 173 μm, d = 100 μm, and w = 62 μm.

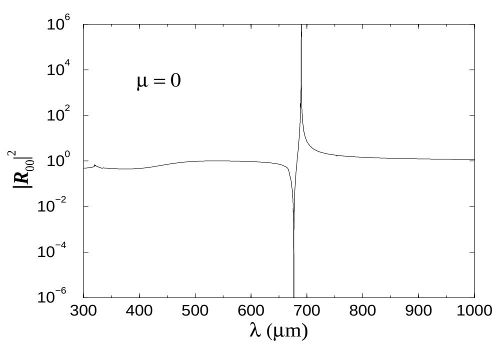

Figure 2: Plot of $|\mathcal{R}_{00}|^2$ as a function of λ.

Figure 2 reveals that there is a singularity at 690 μm preceded by a zero at 677 μm. A singularity in $\mathcal{R}_{00}$ means that the grating supports the zeroth-order outgoing evanescent wave without any incident wave. In order to satisfy the boundary condition, the zeroth-order wave is accompanied by higher spectral order waves in suitable proportions. It can be shown that there is a threshold wavelength $\lambda_{th} = \lambda_g(1+\beta)/\beta$, beyond which all spectral orders are evanescent. For our parameters, λ_{th} = 667 μm; therefore, the singularity appears at a wavelength longer than λ_{th}, and all spectral orders are therefore evanescent. Hence, there is a surface mode comprising of several evanescent spectral orders supported by the grating at this wavelength.

We have also observed that the separation between the location of the singularity and zero reduces as we reduce the groove depth d and vanishes as expected when $d \to 0$, in which case we obtain $\mathcal{R}_{00} = 1$ for all frequencies.

Next, we discuss the behavior around the singularity. We find that around the singularity, $\mathcal{R}_{00}$ has a strong dependence on the growth parameter μ. Figure 3 shows the variation in $1/\mathcal{R}_{00}$ with μ around the singularity. It is clear that near the singularity, $\mathcal{R}_{00}$ can be parametrized as $-i\chi/\mu$ and from the plot, we find χ = 10 per cm.

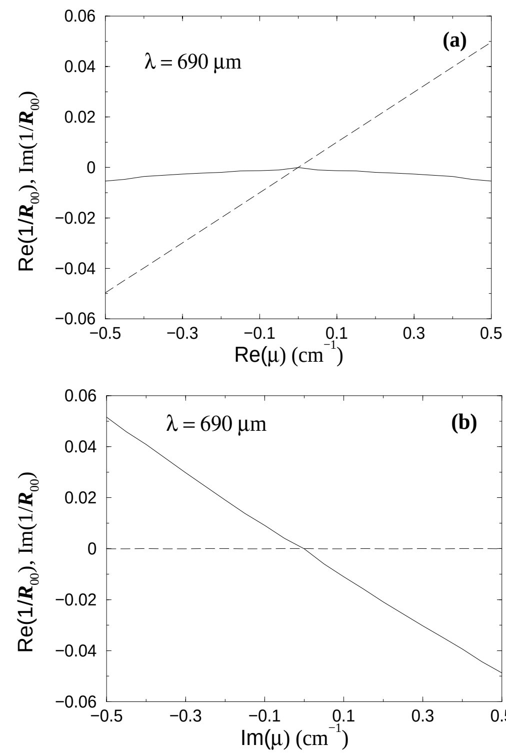

Figure 3: Plots of the real (solid) and imaginary (dashed) part of $1/\mathcal{R}_{00}$ as a function of the real (a) and imaginary (b) part of the growth parameter μ, near the singularity.

We would like to mention that dependence of $\mathcal{R}_{00}$ on μ plays an important role in determining the dispersion relation for the growth rate. Kim et al. [1] have derived the following dispersion relation for the sheet electron having a surface current density K, skimming over the grating at a height h from the top surface of the grating:

$$\mu^2 = \frac{2\pi\Gamma_0}{\gamma^3\beta^3}\frac{K}{I_A}\mathcal{R}_{00}{}^{-2\Gamma_0 h}, \tag{5}$$

where I_A = 17 kA is the Alfven current. Assuming $\mathcal{R}_{00}$ to be a smooth function of λ, they conclude that the dispersion relation is quadratic. However, as explained in the previous paragraph, near the singularity, $\mathcal{R}_{00}$ = $-i\chi/\mu$ and hence we find that the dispersion relation becomes cubic, as also obtained in Refs. [2,7]. However, away from the singularity, $\mathcal{R}_{00}$ has a smooth dependence on μ, and the dispersion relation is quadratic in μ.

0-7803-8859-3/05/$20.00 ©2005 IEEE

FORMULA FOR RESONANT WAVELENGTH

As discussed in the last section, for a given phase velocity βc, the grating supports a surface wave having a free-space resonant wavelength λ_R. It will be useful to have a simple analytic formula for λ_R in terms of β and grating parameters. To the best of our knowledge, such an analytic formula does not exist. Here, we present an approximate formula for λ_R for a shallow rectangular grating. For this purpose, we find that it is more useful to choose the eigenmodes of the groove ($x < 0$) as the basis set for the analysis, unlike the free space modes above the grating ($x > 0$) in the previous sections. Such an analysis is reported by Andrews et al. [2]. They define a scattering matrix $\underline{R}$ that relates the m^{th}-order wave in the groove to the n^{th}-order wave. They find that a surface mode is supported by the grating if the dispersion relation $|\underline{R} - \underline{I}| = 0$ is satisfied where $|\ \ |$ denotes the determinant of the matrix. Solving this dispersion relation numerically, one can calculate λ_R for the given β. Their computation shows that the dispersion relation is accurately described (within a few percent) even if we use a single matrix element R_{00} in the above dispersion relation, implying that we need to simply solve $R_{00} = 1$, which in our notation can be written as

$$R_{00} = -\underbrace{\frac{2k}{w\lambda_g}\tan(kd)}\sum_{m=-1}^{m=1}\frac{\cos(\alpha_m w) - 1}{\Gamma_m \alpha_m^2} = 1. \qquad (6)$$

The above simplified dispersion relation is remarkable since it makes an approximate analytic calculation possible here. This equation can be solved for λ in terms of grating parameters for a given β, and the solution gives us λ_R. Note that here we are considering the EM field supported by the grating in the absence of electron beam and hence, $\mu = 0$. We now try an approximate solution of the above equation for the case $kd \ll \pi/2$ for which the underbraced term will be very small. Hence, the solution of the above equation requires the series sum to be very large. Out of three terms in the series, only the $m = 1$ term can blow up since $\Gamma_1 = 0$ for $\lambda = \lambda_R^0$, where $\lambda_R^0 = \lambda_g(1+\beta)/\beta$. Hence, an approximate solution can be tried around $\lambda = \lambda_R^0$ by neglecting the two terms corresponding to $m = 0$ and -1, and retaining only the $m = 1$ term in Eq. (6). An approximate solution of the resulting equation gives us

$$\lambda_R = \lambda_R^0 + \Delta\lambda, \qquad (7)$$

where $\Delta\lambda = (\lambda_g/2)(\delta/k_R^0)^2$, $k_R^0 = 2\pi/\lambda_R^0$, and δ is given by

$$\delta = \frac{2\tan(k_R^0 d)}{k_R^0 w \lambda_g}\left\{1 - \cos(w k_R^0)\right\}.$$

Note that Eq. (6) actually gives a quadratic equation in λ having two solutions, and we have written only one solution. The second solution is not physical since it is at a wavelength $\lambda < \lambda_{th}$, which means there will be radiating modes at that wavelength, and the grating can not support radiating modes without any external source. Similarly, one can try the solution of Eq. (6) around $\lambda = \lambda_g(1-\beta)/\beta$ (where $\Gamma_1 = 0$), but the solution is not physical.

Figure 4 shows a comparision between values of λ_R calculated from Eq. (7) and results obtained from more rigorous numerical calculations in the previous section for different values of groove depth d. We find that the agreement is good up to $d = 100$ μm, which means our approximate formula is valid up to $kd = 0.9$.

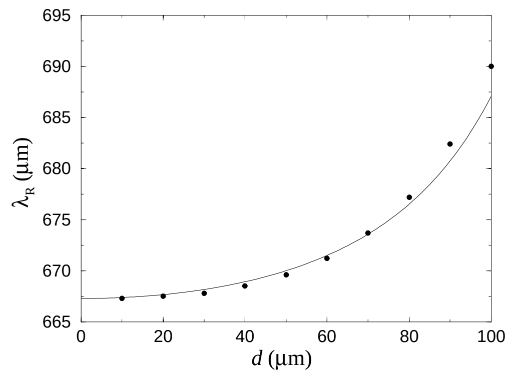

Figure 4: Plot of resonant wavelength λ_R as a function of groove depth d. The solid line is obtained using Eq. (7) and filled circles are obtained by detailed numerical calculation.

CONCLUSIONS

To summarize, we have presented a calculation of $\mathcal{R}_{00}$ for a rectangular grating and studied its frequency dependence. We introduced and evaluated a parameter χ to characterize the behavior of $\mathcal{R}_{00}$ around the singularity. We have also presented an approximate formula for calculating the resonant wavelength of a rectangular grating for the given phase velocity. The analysis is useful in developing a better understanding of SP-FELs [9].

REFERENCES

[1] K.-J. Kim and S. B. Song, Nucl. Instrum. Methods Phys. Res. A 475 (2001) 158.

[2] H. L. Andrews and C. A. Brau, Phys. Rev. ST Accel. Beams, 7 (2004) 07071.

[3] G. Toraldo di Francia, Nuovo Cimento 16 (1960) 61.

[4] Lord Rayleigh (J. W. Strutt), Proc. R. Soc. London Ser. A 79 (1907) 399.

[5] *Electromagnetic Theory of Gratings*, edited by R. Petit (Springer-Verlag, Berlin, 1980).

[6] P. M. Van den berg and T. H. Tan, J. Opt. Soc. Am. 64 (1974) 325.

[7] L. Schachter and A. Ron, Phys. Rev. E 49 (1989) 3340.

[8] J. Urata et al., Phys. Rev. Lett. 80 (1998) 516.

[9] V. Kumar and K.-J. Kim, under preparation.