RADIALLY POLARIZED ION CHANNEL LASER*

R. A. Bosch,# Synchrotron Radiation Center, University of Wisconsin-Madison,
3731 Schneider Dr., Stoughton, WI 53589, USA

Abstract

Radially polarized radiation is amplified by a free electron laser (FEL) in which the undulator is an ion channel with uniform density. For long betatron wavelengths and low gain per pass, the gain at a given distance from the axis is three-eighths the gain of a periodic ion channel laser with the same wiggler parameter. For amplification of short wavelengths by an ultrarelativistic electron beam, a uniform-density ion channel requires a much higher ion density than a periodic ion channel laser.

INTRODUCTION

Consider a cold, uniform, round electron beam propagating in a round ion channel of constant density. When the ion density in the laboratory equals the electron density times $1/\gamma^2$ (where γ is the beam's relativistic factor), force-free beam transport may be obtained [1]. If a beam is injected off-axis, calculations indicate that transverse "betatron" oscillations of the beam centroid may amplify linearly polarized radiation to create an "ion channel laser" [2, 3, 4]. When a mismatched beam is injected on-axis, the beam undergoes radial betatron oscillations that are stationary in the laboratory [5].

We show that radially polarized radiation may be amplified by these radial oscillations, to create a radially polarized ion channel laser. In the low-gain-per-pass, long betatron-wavelength limit, the amplification is three-eighths that of a periodic ion channel laser [6] with the same transverse quiver velocity.

RADIAL MOTION

To model "force" bunching [3, 4, 6], radiation is included in the transverse dynamics. We consider a cold beam with nonrelativistic electron motion in the beam frame. This requires that the ion channel act as an undulator, in which an electron's velocity deviates by less than the angle $1/\beta\gamma$ from the axis, where $\beta > 0$ is the beam velocity divided by the speed of light c.

Consider a stationary ion channel with entrance at $z_{lab} = 0$, whose density in the laboratory frame is $n_{i\text{-}lab}$ for $0 < z_{lab} < L_{lab}$ and $r < r_c$. Here, z_{lab} and r are axial and radial coordinates, while L_{lab} and r_c are the channel's length and radius. In the frame moving with the beam as it enters the undulator, the ion channel density is $n_0 = \gamma_{\|} n_{i\text{-}lab}$, where $\beta_{\|} > 0$ and $\gamma_{\|}$ describe the beam's axial velocity. For $r < r_c$, the radial electric field from the ion channel is

$$E_c(r) = -en_0 r/2\varepsilon_0, \tag{1}$$

where $e < 0$ is the electron charge and ε_0 is the permittivity of free space. The magnetic field in the beam frame is in the azimuthal (ϕ) direction, with $B_c = -\beta_{\|} E_c/c$.

We now consider the amplification of weak radiation when the beam undergoes radial betatron oscillations that are stationary in the laboratory. For a radially polarized wave traveling forward, the radial electric field in the low-gain-per-pass limit is

$$E_r(r,z,t) = E_0(r)\cos(k_r z - \omega_r t + \phi_r). \tag{2}$$

The azimuthal magnetic field B_r equals E_r/c, with wave number $k_r > 0$, phase ϕ_r, and frequency $\omega_r = ck_r > 0$.

Consider a perturbation of the annular beam segment that enters the ion channel when $(z, t) = (0,0)$ and oscillates about equilibrium radius r. When the perturbed axial motion is a uniform drift with nonrelativistic velocity v_0, the radial perturbation $\delta(t)$ obeys

$$\frac{d^2\delta}{dt^2} = \frac{e^2\int_0^{r+\delta} n_e(r',z,t)2\pi r' dr'}{2\pi m\varepsilon_0(r+\delta)} + \frac{e}{m}\left(1+\frac{\beta_{\|}v_0}{c}\right)E_c(r+\delta,z,t) + \frac{e}{m}\left(1-\frac{v_0}{c}\right)E_r(r+\delta,z,t), \tag{3}$$

where m is the electron mass. The first term describes the radial electric field from the beam when the laboratory-frame betatron wavelength greatly exceeds the beam radius. The terms proportional to v_0 result from the magnetic fields B_c and B_r. For laminar flow, $\int_0^{r+\delta} n_e(r',z,t)2\pi r' dr'$ is a constant of motion equaling $(1+\beta_{\|}v_0/c)n_0\pi r^2$. To lowest order in δ, eq. (3) gives

$$\frac{d^2\delta}{dt^2} = \frac{-e^2 n_0}{m\varepsilon_0}\left(1+\frac{\beta_{\|}v_0}{c}\right)\delta + \frac{e}{m}\left(1-\frac{v_0}{c}\right)E_0\cos(k_r z - \omega_r t + \phi_r). \tag{4}$$

(For brevity, we have suppressed the dependence of functions upon r in our notation.) According to eq. (4), the radial velocity of an electron is the sum of a betatron oscillation and forced oscillation from the radiation. The betatron oscillation's radial velocity is

$$v_c(z,t) = -\hat{a}_w c\sin\left[\omega_\beta\left(1+\beta_{\|}v_0/c\right)^{1/2} t + \phi_\beta\right], \tag{5}$$

where $\omega_\beta = (n_0 e^2/m\varepsilon_0)^{1/2}$ is the electron plasma frequency. In the laboratory, the betatron wavelength is $\lambda_{\beta\text{-}lab} = (2\pi\beta_{\|}\gamma_{\|}c/\omega_\beta)(1+v_0/\beta_{\|}c-\beta_{\|}v_0/2c) \approx 2\pi\beta_{\|}\gamma_{\|}c/\omega_\beta$; the betatron frequency is $\omega_{\beta\text{-lab}} = (\omega_\beta/\gamma_{\|})(1+\beta_{\|}v_0/c-\beta_{\|}v_0/2c)^{-1} \approx \omega_\beta/\gamma_{\|} = (n_{i\text{-}lab}e^2/\varepsilon_0 m\gamma_{\|})^{1/2}$. Letting $k_\beta \equiv \beta_{\|}\omega_\beta/2c$ define an effective wavenumber for the betatron oscillations yields

$$v_c(z,t) = -\hat{a}_w c\sin\left(k_\beta z + \omega_\beta t + \phi_\beta\right). \tag{6}$$

Since $\lambda_{\beta\text{-}lab}$ depends upon the electron's axial velocity, $k_\beta \neq \omega_\beta/\beta_{\|}c$. The amplitude $\hat{a}_w$ and phase ϕ_β are determined by the initial conditions of the beam segment (i.e., radius

* Work supported by NSF grant DMR-0084402
# bosch@src.wisc.edu

and radial velocity). Our assumption of nonrelativistic electron velocities in the beam frame requires $\hat{a}_w << 1$.

The forced oscillation from the radiation obeys

$$v_r(z,t) = a_r c \sin(k_r z - \omega_r t + \phi_r), \quad (7)$$

with

$$a_r = \frac{e(1 - v_0/c)\hat{\omega}_r E_0}{mc(\hat{\omega}_\beta^2 - \hat{\omega}_r^2)} . \quad (8)$$

In eq. (8), $\hat{\omega}_r \equiv \omega_r(1 - v_0/c)$ and $\hat{\omega}_\beta \equiv \omega_\beta(1 + \beta_\| v_0/2c)$ are the angular frequencies of the radiation and betatron oscillations experienced by an electron traveling with velocity v_0. Note that $a_r \propto (\hat{\omega}_\beta - \hat{\omega}_r)^{-1}$.

A time-dependent axial velocity may be approximated to arbitrary accuracy by constant-velocity segments. Thus, by letting the velocity v_0 in eq. (8) equal the average axial velocity in the undulator, eqs. (6)–(8) approximately describe small time-dependent axial velocities.

AXIAL MOTION

To describe "inertial" bunching, radiation is included in the axial dynamics [3, 4, 6]. We consider the Compton regime where the axial space-charge forces may be neglected. An electron whose initial axial position z is 0 and radius is r obeys, to lowest order in the radiation field

$$\frac{d^2 z}{dt^2} = \frac{e}{m} v_c B_c + \frac{e}{m} v_c B_r + \frac{e}{m} v_r B_c , \quad (9)$$

where radius $\approx r$ on the right hand side (RHS). The solution with initial conditions $z(0) = dz/dt(0) = 0$ is the sum of three terms describing radiation-independent axial motion, inertial bunching, and force bunching. The radiation-independent motion obeys $d^2 z_0/dt^2 = (e/m) v_c B_c$ where $z \approx v_0 t$ on the RHS of the equation, with v_0 equaling the average axial velocity in the undulator. The solution with initial conditions $z_0(0) = dz_0/dt(0) = 0$ is

$$z_0(t) = \frac{e^2 n_0 \beta_\| \hat{a}_w r}{2m\varepsilon_0 \hat{\omega}_\beta^2}\left[\sin(\hat{\omega}_\beta t + \phi_\beta) - \hat{\omega}_\beta t \cos\phi_\beta - \sin\phi_\beta\right] . \quad (10)$$

Equation (10) gives the average axial velocity as

$$v_0 \approx \frac{-e^2 n_0 \beta_\| \hat{a}_w r \cos\phi_\beta}{2m\varepsilon_0 \hat{\omega}_\beta} \approx \frac{-\omega_\beta \beta_\| \hat{a}_w r \cos\phi_\beta}{2} . \quad (11)$$

Our assumption of nonrelativistic velocities in the beam frame requires that the plasma skin depth in the beam frame $c/\omega_\beta >> \beta_\| \hat{a}_w r/2$. In the laboratory, this requirement is $\lambda_{\beta\text{-}lab} >> \pi\beta_\|^2 \hat{a}_w \gamma_\| r$.

The inertial bunching term [3, 4, 6] results from the axial radiation force on an electron, obeying $d^2 z_i/dt^2 = (e/m) v_c B_r$ where $z \approx z_0(t)$ on the RHS. For $\hat{a}_w << 1$, approximating $z_0(t) \approx v_0 t$ on the RHS for the fundamental FEL mode gives the solution with initial conditions $z_i(0) = dz_i/dt(0) = 0$:

$$z_i(t) = \frac{e\hat{a}_w E_0}{2m}\left[\frac{\sin(\omega_+ t + \phi_\beta - \phi_r) - \sin(\phi_\beta - \phi_r) - \omega_+ t \cos(\phi_\beta - \phi_r)}{\omega_+^2} + \frac{\sin(\omega_- t + \phi_\beta + \phi_r) - \sin(\phi_\beta + \phi_r) - \omega_- t \cos(\phi_\beta + \phi_r)}{\omega_-^2}\right] \quad (12)$$

where $\omega_+ \equiv \hat{\omega}_\beta + \hat{\omega}_r$ and $\omega_- \equiv \hat{\omega}_\beta - \hat{\omega}_r$. For effective amplification of radiation, $|\omega_-| << \omega_+$, so that

$$z_i(t) = \frac{e\hat{a}_w E_0}{2m\omega_-^2}\left[\sin(\omega_- t + \phi_\beta + \phi_r) - \sin(\phi_\beta + \phi_r) - \omega_- t \cos(\phi_\beta + \phi_r)\right] . \quad (13)$$

The force bunching term [3, 4, 6] results from the transverse radiation force on an electron, obeying $d^2 z_f/dt^2 = (e/m) v_r B_c$ where $z \approx z_0(t)$ on the RHS. For $\hat{a}_w << 1$, approximating $z_0(t) \approx v_0 t$ on the RHS for the fundamental FEL mode gives the solution with initial conditions $z_f(0) = dz_f/dt(0) = 0$:

$$z_f(t) = \frac{\beta_\| e^2 n_0 a_r r}{2m\varepsilon_0 \hat{\omega}_r^2}\left[\sin(\hat{\omega}_r t - \phi_r) + \sin\phi_r - \hat{\omega}_r t \cos\phi_r\right] . \quad (14)$$

In contrast to a periodic ion channel laser [6], the inertial bunching term does not equal that from force bunching for $\hat{a}_w << 1$.

GAIN

The change in an electron's energy from interaction with the radiation obeys

$$\frac{d\varepsilon}{dt} = e v_r E_r + e v_c E_r , \quad (15)$$

where v_r, v_c and E_r are evaluated at radius r and the axial position $z(t)$ calculated in the previous section. The change in an average electron's energy is given by averaging over the phase of the radiation ϕ_r. To order E_0^2, the first term on the RHS does not contribute to this average. When motion in the beam frame is nonrelativistic ($c/\omega_\beta >> \beta_\| \hat{a}_w r/2$), eq. (15) becomes

$$\begin{aligned} \langle d\varepsilon/dt\rangle_{\phi_r} &= \langle e v_c E_r\rangle_{\phi_r} \approx \\ &\langle z\cos\phi_r\rangle_{\phi_r}\left[\frac{-e\hat{a}_w c E_0}{2}\left[k_+ \cos(\omega_- t + \phi_\beta) + k_- \cos(\omega_+ t + \phi_\beta)\right]\right] \\ &+ \langle z\sin\phi_r\rangle_{\phi_r}\left[\frac{e\hat{a}_w c E_0}{2}\left[k_+ \sin(\omega_- t + \phi_\beta) - k_- \sin(\omega_+ t + \phi_\beta)\right]\right], \end{aligned} \quad (16)$$

where $k_+ \equiv k_\beta + k_r$ and $k_- \equiv k_\beta - k_r$. For inertial bunching, eq. (13) gives

$$\begin{aligned} \langle z_i\cos\phi_r\rangle_{\phi_r} &= \frac{e\hat{a}_w E_0}{4m\omega_-^2}\left[\sin(\omega_- t + \phi_\beta) - \sin\phi_\beta - \omega_- t\cos\phi_\beta\right] \\ \langle z_i\sin\phi_r\rangle_{\phi_r} &= \frac{e\hat{a}_w E_0}{4m\omega_-^2}\left[\cos(\omega_- t + \phi_\beta) - \cos\phi_\beta + \omega_- t\sin\phi_\beta\right]. \end{aligned} \quad (17)$$

For force bunching, eq. (14) gives

$$\begin{aligned} \langle z_f\cos\phi_r\rangle_{\phi_r} &= \frac{\beta_\| e^2 n_0 a_r r}{4m\varepsilon_0\hat{\omega}_r^2}(\sin\hat{\omega}_r t - \hat{\omega}_r t) \\ \langle z_f\sin\phi_r\rangle_{\phi_r} &= \frac{-\beta_\| e^2 n_0 a_r r}{4m\varepsilon_0\hat{\omega}_r^2}(\cos\hat{\omega}_r t - 1). \end{aligned} \quad (18)$$

Let $\Delta\varepsilon \equiv \int_0^T \langle d\varepsilon/dt\rangle_{\phi_r} dt$ be the average energy change per electron from interacting with radiation. Here, T is the ion channel transit time in the beam frame. The average energy change $\Delta\varepsilon$ is the sum of the contributions from inertial and force bunching. For $|\omega_-| << \omega_+$, eqs. (16) and (17) give the inertial bunching contribution

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$$\Delta\varepsilon_{\rm i} = \frac{-e^2 E_0{}^2 \hat{a}_w{}^2 ck_+ T^3}{8m}\left(\frac{2-2\cos\omega\_T - \omega\_T\sin\omega\_T}{\omega\_{}^3 T^3}\right), \quad (19)$$

while eqs. (16) and (18) give the force bunching contribution

$$\Delta\varepsilon_{\rm f} = \frac{-e^4 E_0{}^2 \hat{a}_w \beta_\| n_0 r k_+ T^3}{16m^2\varepsilon_0\omega_r(\omega\_T)^3}\left[\begin{array}{l}\cos\phi_\beta(1-\cos\omega\_T - \omega\_T\sin\omega\_T) \\ +\sin\phi_\beta(\sin\omega\_T - \omega\_T\cos\omega\_T)\end{array}\right]. \quad (20)$$

In the beam frame, the energy transferred to the forward wave within a transverse area A during a time t_0 is $-n_0 A\beta_\| ct_0\Delta\varepsilon$ for $v_0 << c$. The time-averaged Poynting vector of the radiation is $< S > = \varepsilon_0 cE_0{}^2/2$, with energy density $< S >/c$. Since the relative velocity between the forward wave and undulator is $(1+\beta_\|)c$, the electromagnetic energy passing through the undulator is $(< S >/c)(1+\beta_\|)ct_0 A$. The radiation energy gain per pass at radius r therefore obeys

$$\text{gain} = \frac{-n_0 A\beta_\| ct_0\Delta\varepsilon}{\langle S\rangle(1+\beta_\|)\,t_0 A} = \left(\frac{-2\beta_\|}{1+\beta_\|}\right)\frac{n_0\Delta\varepsilon}{\varepsilon_0 E_0{}^2}. \quad (21)$$

The gain is the sum of the gain from inertial bunching and the gain from force bunching. For $\gamma >> 1$, the gain from inertial bunching is given by eqs. (19) and (21)

$$\text{gain}_{\rm i} = \frac{n_0 e^2 k_+ c\hat{a}_w{}^2 T^3}{8m\varepsilon_0}\left(\frac{2-2\cos\omega\_T - \omega\_T\sin\omega\_T}{\omega\_{}^3 T^3}\right). \quad (22)$$

In the laboratory frame, the maximum transverse velocity divided by c is obtained from the radial and axial velocities in the beam frame when $|v_c|$ is largest:

$$\beta_{\perp-lab} = \frac{a_w}{\gamma} \approx \frac{\hat{a}_w}{\gamma_\|[1-\beta_\|{}^2\hat{a}_w r\omega_\beta\cos\phi_\beta/2c]} \approx \frac{\hat{a}_w}{\gamma}, \quad (23)$$

where $a_w \approx \hat{a}_w$ is the wiggler parameter. For $\gamma >> 1$, $k_+ = k_\beta + k_r = 1.5\gamma_\|\omega_{\beta\text{-lab}}/c$, where $\omega_{\beta\text{-lab}}$ is the laboratory-frame betatron frequency. Thus, the gain from inertial bunching at radius r is given to lowest order in the wiggler parameter a_w as

$$\text{gain}_{\rm i} = \frac{3n_{e-lab}\,e^2\omega_{\beta-lab}\,a_w{}^2 L_{lab}{}^3}{16m\varepsilon_0\,c^3\gamma^3}\left[\frac{2-2\cos\omega\_T - \omega\_T\sin\omega\_T}{(\omega\_T)^3}\right], \quad (24)$$

where $n_{e\text{-lab}}$ is the e-beam density and L_{lab} is the ion channel's length measured in the laboratory frame.

For $\gamma >> 1$, the gain at radius r from force bunching is, to lowest order in a_w,

$$\text{gain}_{\rm f} = \frac{3a_w\,r\,\gamma\,\omega_{\beta-lab}{}^4 L_{lab}{}^3}{32c^4(\omega\_T)^3}\left[\begin{array}{l}\cos\phi_\beta(1-\cos\omega\_T - \omega\_T\sin\omega\_T) \\ +\sin\phi_\beta(\sin\omega\_T - \omega\_T\cos\omega\_T)\end{array}\right]. \quad (25)$$

Note that $\text{gain}_{\rm f}$ depends upon the amplitude and phase of the betatron oscillations, while $\text{gain}_{\rm i}$ depends only upon their amplitude.

Here,

$$\omega\_T = [\omega_{\beta-lab}(1+v_0/2c) - \omega_{r-lab}(1-v_0/c)/2\gamma^2]T_{lab}, \quad (26)$$

where $T_{lab} = L_{lab}/\beta_\| c$ is the ion-channel transit time and $\omega_{r\text{-lab}}$ is the radiation's angular frequency in the laboratory. For an ultrarelativistic beam undergoing many betatron oscillations, maximum gain occurs for

$$\omega_{r-lab} \approx \frac{2\gamma^2\omega_{\beta-lab}}{1-3v_0/2c} \approx 2\gamma^2\omega_{\beta-lab}. \quad (27)$$

For $\gamma >> 1$, $|\text{gain}_{\rm f}/\text{gain}_{\rm i}|$ is $\sim r\gamma\omega_{\beta\text{-lab}}/2ca_w$, so that the gain from inertial bunching dominates when $a_w >> \pi r\gamma/\lambda_{\beta\text{-lab}}$. Thus, for a wiggler parameter comparable to unity, inertial bunching dominates for long betatron wavelengths obeying $\lambda_{\beta\text{-lab}} >> \pi r\gamma$. In this case, the gain is 3/8 times the gain of a periodic ion channel laser [6] or planar magnetostatic undulator with the same quiver velocity, while the wavelength experiencing maximum gain is modified by the value of v_0 in eq. (27). The gain is smaller than that of a periodic ion channel laser for two reasons. First, force bunching is small for long betatron wavelengths, while it nearly equals the inertial bunching in a periodic ion channel laser. Second, the laboratory betatron wavelength depends upon the beam energy, so that $k_\beta = \beta_\|\omega_\beta/2c < \omega_\beta/\beta_\| c$. This reduces the gain from inertial bunching [2, 3, 4].

When $\gamma >> 1$ and $\lambda_{\beta\text{-lab}} < \pi r\gamma$, the above formulas apply when the beam-frame velocities are nonrelativistic ($a_w << \lambda_{\beta\text{-lab}}/\pi r\gamma$), in which case force bunching dominates.

DISCUSSION

A constant-density ion channel, in which the phase-mix damping of betatron oscillations is negligible, may be used to amplify radially polarized radiation. For an ultrarelativistic beam with a long betatron wavelength, the gain is three-eighths times the gain of a periodic ion channel laser with the same wiggler parameter.

A periodic ion channel with wiggler parameter comparable to unity and period $\lambda_{w\text{-lab}}$ equaling ten times the beam radius requires an ion density in the laboratory of $\sim 40\pi\varepsilon_0 mc^2/e^2\lambda_{w\text{-lab}}{}^2$ for $\gamma >> 3$ [6]. A constant-density ion channel with $\lambda_{\beta\text{-lab}}$ equaling ten times the beam radius requires an ion density that is a factor of $\pi\gamma/10$ times as large. Thus, for ultrarelativistic beams, amplification of short wavelengths may be achieved with a much smaller ion density if a periodic ion density channel is used.

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