TRANSIENT RESISTIVE WALL WAKE FOR VERY SHORT BUNCHES*

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Abstract

The catch up distance for the resistive wall wake in a round pipe is approximately equal to the square of the pipe radius divided by the bunch length. The standard formulae for this wake are applicable at distances much larger than the catch up distance. In this paper, we calculate the resistive wall wake at distances compared with the catch up distance assuming a constant wall conductivity.

INTRODUCTION

A recent proposal to generate attosecond [1] X-ray pulses using energy modulation of the beam via resonant interaction with an optical laser involves propagation of relativistic electron bunches as short as ~ 30 nm. Calculation of the short range resistive wave generated by such a bunch involves several issues not encountered in the classical theory of wakefields.

First, one cannot make the usual assumption $v = c$ for such bunches and typical transverse sizes b of the vacuum chamber. Indeed, this assumption is valid if the spot size on the wall of the relativistically contracted vacuum field of a point charge, b/γ, is much smaller that the bunch length σ_z. For b equal to several millimeters and $\gamma \sim 10^4$ this, however, is not valid if $\sigma_z \sim 30$ nm. Second, one cannot assume a constant conductivity σ and has to take into account the frequency dependence $\sigma(\omega)$ [2]. Moreover, the anomalous skin effect may play a role at such high frequencies. Finally, the catch up distance for the wake, which is estimated as $b^2/2\sigma_z$ is of the order of tens of meters, and the usual definition of the wake as due to the steady state fields in the pipe is not valid. One has to take into account the transient effect of the field build up as the bunch propagates from the entrance to the exit of the pipe.

A complete calculation of the wake which realistically takes into account all of the above features of the beam interaction with the wall is a complicated problem. In this paper we will address only one part of it. Specifically, we will study the time dependent build up of the wakefield assuming a constant conductivity σ of the pipe and also using the $v = c$ assumptions. The method developed in this paper can be generalized to include the more realistic assumptions about the beam-wall interaction.

FORMULATION OF THE PROBLEM

Consider geometry shown in Fig. 1. An infinitely long round pipe of radius b has wall conductivity σ in the right part, $z > 0$, and an infinite conductivity at $z < 0$. A particle beam travelling with the speed close to the speed of light from left to right enters the region $z > 0$ at time $t = 0$. Our goal is to calculate how the wakefield builds up when the beam propagates through the pipe. We will constrain our analysis by studying the longitudinal wake only.

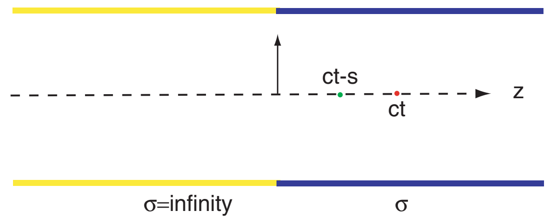

Figure 1: A round pipe of radius b has wall conductivity σ at $z > 0$; the wall conductivity at $z < 0$ is infinite. Two point charges separated by the distance s propagate along the axis of the pipe with the speed of light c.

As always in the wakefield theory, we will consider 2 particles of the beam separated by distance s on the axis of the pipe. The electric field E_z generated by the leading particle will act on the trailing one.

In the left part of the pipe, $z < 0$, where the wall has an infinite conductivity, the electric and magnetic fields of the leading particle are

$$E_r^{(\mathrm{vac})}(z,r,t) = B_\theta^{(\mathrm{vac})}(z,r,t) = \frac{2q}{r}\delta(z-ct)\,. \tag{1}$$

The longitudinal component of the field is zero at $z < 0$. The superscript "vac" indicates that those fields are the same as in free space.

For the finite conductivity part of the pipe, the boundary condition on the surface of the wall is given by the so called Leontovich boundary condition [3]

$$\hat{E}_z^{(\mathrm{wall})}(z,\omega) = (i-1)\sqrt{\frac{\omega}{8\pi\sigma}}\hat{B}_\theta^{(\mathrm{wall})}(z,\omega)\,, \tag{2}$$

where the hat denotes the Fourier transform, $\hat{f}(\omega) = (2\pi)^{-1}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$. The total field in this part of the pipe is equal to the vacuum field given by Eq. (1) and a non-singular solution of Maxwell's equations which satisfies the boundary condition

$$E_z|_{r=b}(z,t) = E_z^{(\mathrm{wall})}(z,t)\,. \tag{3}$$

Only the latter contributes to the wake.

We will use a perturbation approach to calculate E_z at $z > 0$. In this approach, the magnetic field of the leading particle on the wall is assumed to be equal to its value in the pipe of infinite conductivity,

$$B_\theta^{(\mathrm{wall})} = B_\theta^{(\mathrm{vac})}|_{r=b}\,. \tag{4}$$

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Note that this approach gives the correct steady state wake in the limit $s \gg (b^2 c/4\pi\sigma)^{1/3}$ [4].

SOLUTION OF MAXWELL'S EQUATIONS

As a first step, let us calculate explicitly the time dependent function $E_z^{(\mathrm{wall})}(z,t)$. For $\hat{B}_\theta^{(\mathrm{wall})}(z,\omega)$ we have

$$\hat{B}_\theta^{(\mathrm{wall})}(z,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt B_\theta^{(\mathrm{vac})}(z,b,t)e^{i\omega t} = \frac{q}{\pi bc}e^{i\omega z/c}\,. \tag{5}$$

Using Eq. (2) we find

$$\begin{aligned} E_z^{(\mathrm{wall})}(z,t) &= \int_{-\infty}^{\infty} d\omega \hat{E}_z^{(\mathrm{wall})}(z,\omega)e^{-i\omega t} \\ &= (i-1)\frac{q}{\pi bc}\sqrt{\frac{\omega}{8\pi\sigma}}\int_{-\infty}^{\infty} d\omega\sqrt{\omega}e^{-i\omega t + i\omega z/c} \\ &= \frac{q}{2\pi bc\sigma^{1/2}}\frac{h(t-z/c)}{(t-z/c)^{3/2}}\,, \qquad \text{for } z>0 \end{aligned} \tag{6}$$

where $h(t)$ is the step function. The integral $\int_{-\infty}^{\infty} d\omega\sqrt{\omega}e^{-i\omega t + i\omega z/c}$ in the last equation is computed by shifting the integration path in the complex plane ω (see, e.g., Ref. [5]).

To find $E_z(z,r,t)$ we need to solve the wave equation

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial E_z}{\partial r} + \frac{\partial^2 E_z}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} = 0 \tag{7}$$

with the boundary condition at $r = b$ given by Eq. (6). We will use the Green function method and define a Green function $G(z,t) = \tilde{E}_z(z,0,t)$, where $\tilde{E}_z(z,r,t)$ is the solution of Eq. (7) with the boundary condition $\tilde{E}_z(z,b,t) = \delta(z)\delta(t)$. The Green function gives the longitudinal field on the axis generated by a localized source on the wall turned on for an infinitely short period of time ($\propto \delta(z)\delta(t)$). If the Green function is known, then the longitudinal field on the axis generated by the boundary condition Eq. (6), which we denote by $\mathcal{E}(z,t)$, is given by the following expression:

$$\mathcal{E}(z,t) = \int_0^{\infty} dz_0 \int_{-\infty}^{\infty} dt_0 G(z-z_0, t-t_0)E_z^{(\mathrm{wall})}(z,t)\,.$$

To find $\tilde{E}_z(z,r,t)$ we will use the Hertz potential $\Pi(z,r,t)$, related to the electric field through the following equation [6]:

$$\tilde{E}_z(z,r,t) = \left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Pi(z,r,t)\,. \tag{8}$$

Using Laplace transformation, one can find the following expression for Π,

$$\Pi(z,r,t) = \sum_{m=1}^{\infty}\frac{cJ_0(\mu_m r/b)}{\mu_m J_1(\mu_m)} \times J_0\left(\frac{\mu_m c}{b}\sqrt{t^2 - \frac{z^2}{c^2}}\right)h\left(t - \frac{|z|}{c}\right)\,, \tag{9}$$

where J_0 is the Bessel function of zeroth order. The Hertz potential on the axis which we denote by $\Pi_0(z,t)$, $\Pi_0(z,t) = \Pi(z,0,t)$, is obtained from Eq. (9) by replacing $J_0(\mu_m r/b)$ with 1. The plot of the function Π_0 is shown in Fig. 2. It is seen from this plot, that $\Pi_0 = 1/2$ for

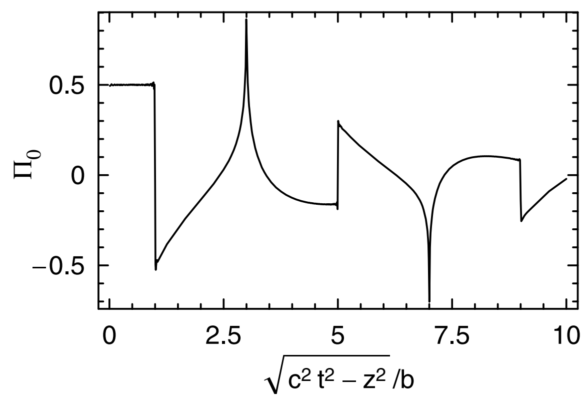

Figure 2: Plot of function Π_0 versus $\sqrt{t^2c^2 - z^2}/b$.

$\sqrt{t^2c^2 - z^2} < b$, which means that the electric field on axis given by Eq. (8) is equal to zero for $ct < \sqrt{z^2 + b^2}$. Of course, this is exactly what one expects from the causality argument.

Introducing a new variable $s = ct - z$, we can express G as

$$G(z,t) = \frac{1}{c}\left(\frac{\partial}{\partial t}\right)_s\left(\frac{\partial \Pi_0}{\partial z} - \frac{1}{c}\frac{\partial \Pi_0}{\partial t}\right)\,, \tag{10}$$

where $(\partial/\partial t)_s = c\partial/\partial z + \partial/\partial t$.

TIME DEPENDENT WAKE

We now define the longitudinal transient wake as

$$w_l(s,t) = -\frac{c}{q}\int_{-\infty}^{t} dt' \mathcal{E}(s,t')\,. \tag{11}$$

Note that this wake does not only depend on the distance s between the particles but also on the time t, or, equivalently, the position inside the pipe $z = ct$. We expect that when $t \to \infty$, the above wake will approach the usual steady-state wake. Using Eq. (9) we can obtain the following expression for the wake

$$\begin{aligned} w_l(s,t) &= -\frac{c}{q}\int_{-\infty}^{t} dt' \int dz_0 dt_0 \\ &\quad \times G(z-z_0, t'-t_0)|_{z=ct'-s} E_z^{(\mathrm{wall})}(z_0,t_0) \\ &= -\frac{1}{q}\int_{-\infty}^{t} dt' \int dz_0 dt_0 \\ &\quad \times \left(\frac{\partial}{\partial t'}\right)_s\left(\frac{\partial \Pi_0}{\partial z} - \frac{1}{c}\frac{\partial \Pi_0}{\partial t'}\right)E_z^{(\mathrm{wall})}(z_0,t_0) \\ &= -\frac{1}{q}\int dz_0 dt_0 H(z-z_0, t-t_0)|_{z=ct-s}E_z^{(\mathrm{wall})}(z_0,t_0)\,, \end{aligned} \tag{12}$$

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with $H(z,t) = (\partial/\partial z - c^{-1}\partial/\partial t)\Pi_0(z,t)$. Note that Π_0 is actually a function of a single argument $\xi = \sqrt{t^2 - z^2/c^2}$, $\Pi_0(z,t) = F(\xi)$.

We are interested in values of z and ct of the order of the pipe length L, and hence, $\xi \sim L/c$. A typical value of s is of the order of the bunch length σ_z, $s \sim \sigma_z$. Assuming $L \gg \sigma_z$, we have $\xi \approx \sqrt{2ts/c}$. Using the following transformation,

$$\left(\frac{\partial}{\partial z} - \frac{1}{c}\frac{\partial}{\partial t}\right) F(\xi) = F'(\xi)\left(\frac{\partial \xi}{\partial z} - \frac{1}{c}\frac{\partial \xi}{\partial t}\right)$$
$$\approx -\sqrt{\frac{2t}{cs}} F'(\xi) \approx -2\left(\frac{\partial F}{\partial s}\right)_t , \qquad (13)$$

we introduce the *wake potential* $W(s,t)$ such that

$$w_l(s,t) = \frac{\partial W(s,t)}{\partial s} , \qquad (14)$$

and find that

$$W(s,t) = \frac{2}{q}\int dz_0 dt_0 \qquad (15)$$
$$\times\, \Pi_0(z - z_0, t - t_0)|_{z=ct-s} E_z^{(\mathrm{wall})}(z_0, t_0)$$
$$= \frac{1}{\pi b c \sigma^{1/2}}\int_0^\infty dz_0 \int_{z_0/c}^\infty dt_0 \frac{\Pi_0(z - z_0, t - t_0)|_{z=ct-s}}{(t - z/c)^{3/2}} .$$

The integration in Eq. (15) can be carried out analytically, with the result $W(s,t) = W_0(s)R(v)$, where

$$W_0(s) = \frac{c^{3/2} t}{\pi b \sqrt{\sigma s}} , \qquad (16)$$

$$R(v) = 1 - 4\sum_{m=1}^{\infty} \frac{\mu_m v \sin(\mu_m v) + \cos(\mu_m v) - 1}{\mu_m^3 v^2 J_1(\mu_m)} ,$$

and $v = \sqrt{2cts/b^2}$. The plot of the function R is shown in Fig. 3; it approaches 1 in the limit $v \to \infty$. Note that in

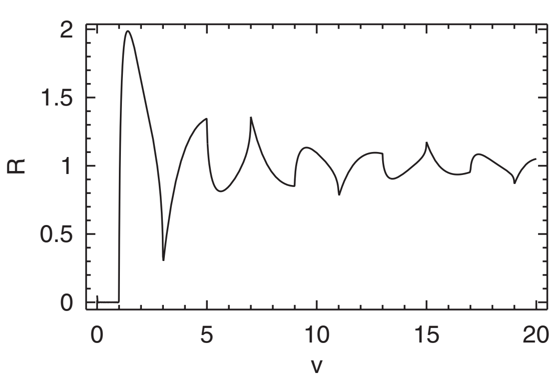

Figure 3: Plot of the function R.

this limit $w_l = \partial W_0/\partial s$, which recovers the resistive wall wake result for an infinitely long pipe.

Using the above result for the time-dependent point charge wake we calculated the wake for a Gaussian bunch. The wake is defined as

$$w_{\mathrm{bunch}}(s,l) = \frac{1}{l}\int_s^\infty ds' \lambda(s') w_l(s' - s, l/c) , \qquad (17)$$

where $\lambda(s) = (1/\sqrt{2\pi}\sigma_z)e^{-s^2/2\sigma_z^2}$ is a Gaussian distribution function with the rms bunch length σ_z. The quantity w_{bunch} has a meaning of the wake accumulated over the distance l of the resistive part of the pipe normalized by the length of the path; it has a dimension of V/C/m. When $l \to \infty$, the wake $w_{\mathrm{bunch}}(s,l)$ should approach to the known result for an infinitely long pipe.

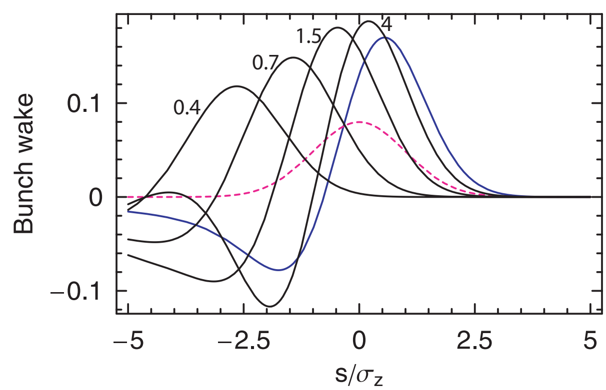

Figure 4: Wake for a Gaussian bunch for different values of the parameter $2l\sigma_z/b^2$ indicated by a number near each curve. The blue curve shows the wake in the limit $l = \infty$, and the dashed magenta line is the Gaussian distribution of the bunch.

The result is shown in Fig. 4, where the vertical axis shows the normalized wake $w_{\mathrm{bunch}}(s,l)(b^2\sigma_z^3\sigma/c)^{1/2}$. The figure clearly demonstrates how the wake accumulates over the distance $\sim b^2/\sigma_z$ which is of the order of the catch up distance.

In conclusion, we calculated a time-dependent resistive wall wake for a simplest model that assumes a constant conductivity and uses a perturbation approach with the assumption $v = c$. Although this models is not directly applicable to the extremely short bunches envisioned in some modern applications, it illustrates the process of wake build after the bunch enters the pipe with resistive walls.

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