

TRANSVERSE INSTABILITY OF A RECTANGULAR BUNCH *

V. Balbekov, FNAL, Batavia, IL 60510, USA

Abstract

Transverse instability of a rectangular bunch is investigated. Known theory of bunched beam instability is modified to take into account 100% spread of synchrotron frequency. Series of equations adequately describing the instability is derived and solved analytically and numerically. The theory is applied to the Fermilab Recycler Ring.

INTRODUCTION

Transverse beam instability – probably resistive wall type – is observed sometimes in the Fermilab Recycler Ring [1, 2]. A damper is designed now to avoid problems at planned increase of the Recycler intensity. It should be taken into account that, typically, the Recycler beam is bunched longitudinally by a barrier-bucket rf waveform. Therefore the bunch has almost rectangular shape, synchrotron frequency is rather low and has 100% spread. One would expect that instability of such beam is a cross between a coasting beam and “normal” bunched beam. Theoretical investigation of this problem and development of requirements to the damper is the main subject of this paper. In doing so, we neglect moderate effects of penetration of particles into the barriers, and nonlinearity of betatron oscillations.

BASIC EQUATIONS

Let us consider transverse dipole moment of a beam $D(\theta) \exp(-i\omega t)$ where θ is azimuth in a rest frame. Its Fourier harmonics satisfy the set of equations (see e.g. [3]):

$$D_k = \frac{i r_0 \omega_0 N}{2\pi \gamma Q_0 Z_0} \sum_{k'} C_{k,k'} D_{k'} Z_k^\perp \quad (1)$$

$$C_{k,k'} = \sum_m \int_0^\infty \frac{I_{m,k}(\epsilon) I_{m,k'}^*(\epsilon) F(\epsilon) d\epsilon}{\omega + \omega_0 Q_0 - m\Omega} \quad (2)$$

where N is the beam intensity, ω_0 and Q_0 are average angular velocity and betatron frequency, ϵ and $\Omega(\epsilon)$ – action and frequency of synchrotron oscillations, $F(\epsilon)$ – normalized distribution function, Z_k^\perp – transverse beam coupling impedance [4], $r_0 = e^2/mc^2 \simeq 1.535 \times 10^{-16}$ cm, $Z_0 = 4\pi/c \simeq 376.7$ Ohm. General formula for form-factors $I_{m,k}$ is

$$I_{m,k}(\epsilon) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{im\phi - i\lambda\theta(\epsilon,\phi)} d\phi \quad (3)$$

where $\lambda = k - Q_0 + \xi/\eta$, ξ and η are chromaticity and slippage factor, and azimuth θ is presented as a function of action and phase of synchrotron oscillations.

In a rectangular potential wall of the length θ_0 , $\theta = 2B|\phi|$, $\epsilon = 4\pi B|p - p_0|$, and

$$\Omega = \frac{\omega_0 |\eta(p - p_0)|}{2p_0 B} = \frac{\omega_0 \epsilon |\eta|}{8\pi p_0 B^2} \quad (4)$$

where p and p_0 are the particle momentum and central one, and $B = \theta_0/2\pi$ is a bunch factor. Then formula (3) gives:

$$I_{m,k} = \frac{\psi_{m,k} + \pi m/2}{\psi_{m,k} + \pi m} \frac{\sin \psi_{m,k}}{\psi_{m,k}} e^{i\psi_{m,k}} \quad (5)$$

where $\psi_{m,k} = \pi(B\lambda - m/2)$. The sum over m in Eq. (2) is calculated analytically in this case providing general formula for the matrix $C_{k,k'}$. However, it is very cumbersome; therefore we will restrict our consideration to the required specific cases.

LOW SYNCHROTRON FREQUENCY

As the first approximation, consider the case when both η and $\Omega \rightarrow 0$. It is reasonable because typical synchrotron frequency in the Recycler is several Hz whereas the space charge tune shift can reach several hundred Hz. The result is: ¹

$$C_{k,k'} = N_{k-k'} \int_{-\infty}^{\infty} \frac{\mathcal{F}(p) dp}{\omega + \omega_0 Q(p)} \quad (6)$$

where $\mathcal{F}(p)$ is normalized distribution function on momentum, $Q(p) = Q_0 + \xi(p - p_0)/p_0$ is the momentum dependent tune, and

$$N_l = \frac{\sin(\pi l B)}{\pi l B} e^{i\pi l B} \quad (7)$$

is Fourier harmonic of normalized beam density which is $1/B$ in the potential wall. Therefore dispersion equation of the instability can be written in the form:

$$1 = \frac{i r_0 \omega_0 N Z^{(ef)}}{2\pi \gamma Q_0 Z_0} \int_{-\infty}^{\infty} \frac{\mathcal{F}(p) dp}{\omega + \omega_0 Q(p)} \quad (8)$$

what is quite similar to the equation for a coasting beam at $\eta = 0$. The only distinction is a replacement of the beam coupling impedance Z_k^\perp by an effective impedance $Z^{(ef)}$ which is defined here as any eigenvalue of the problem:

$$Z^{(ef)} D_k = \sum_{k'} N_{k-k'} Z_{k'}^\perp D_{k'} \quad (9)$$

At $B = 1$, the distinction entirely disappears because $Z^{(ef)} = Z_k^\perp$ at $N_{k-k'} = \delta_{k,k'}$. Some results of numerical solution at $B \neq 1$ are represented below.

* Work supported by the U. S. Department of Energy under contract DE-AC02-76CH03000.

¹Note that it would be wrong simply to take $\Omega = 0$ in Eq. (2) because parameter $\lambda \rightarrow \infty$ at the same time.

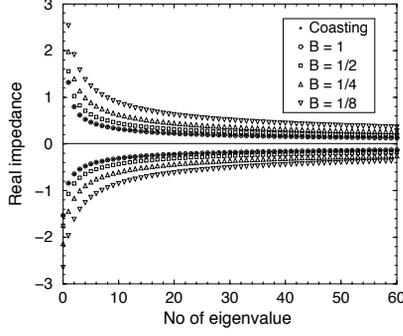


Figure 1: Real part of the effective resistive wall impedance

Resistive wall impedance

First example considered is resistive wall impedance:

$$Z_k^\perp = Z^{(rw)} \frac{\text{sgn}(\kappa) - i}{\sqrt{|\kappa|}} \quad (10)$$

where $\kappa = k + \omega/\omega_0 \simeq k - Q_0$ is normalized frequency in the laboratory frame [4]. Real parts of the eigenvalues are presented in Fig. 1 at $Z^{(rw)}=1$, $Q_0 = 0.425$, and different B . They are arranged in order of decreasing absolute value, what in practice is the same as increasing number of central harmonic of the spectrum. At $B = 1$, the numerical solution coincides with analytical solution of a coasting beam. Note that approximate dependence of varied eigenvalues on bunch factor is between $B^{-0.3} - B^{-0.5}$.

Exponential wake

Another important case is the impedance

$$Z_k^\perp = \frac{Z^{(ex)}}{1 - i\alpha\kappa} \quad (11)$$

corresponding an exponentially decreasing wake field. It turns out that in this case $Z_k^{(ef)} = Z_k^\perp$ regardless the bunch factor, though the eigenfunctions essentially depend on B (the result is obtained both analytically and numerically).

Resistive wall and damper

Combining Eq. (10) and (11) with real negative $Z^{(ex)}$, one can get a model of an accelerator with a resistive wall and simplest wide-band damper:

$$Z_k^\perp = Z^{(rw)} \left(\frac{\text{sgn}(\kappa) - i}{\sqrt{|\kappa|}} - \frac{g}{1 - i\alpha\kappa} \right) \quad (12)$$

Solution at $Q_0=0.425$, $Z^{(rw)}=1$, $\alpha=0.04$, $g = 1.32$ ² is illustrated by Fig. 2 where the eigenvalues multiplied on B are placed in complex plain. It is seen that their absolute value increases at the bunching not faster than $1/B$. It is seen also that $\text{Re } Z^{(ef)} > 0$ for higher modes what means that the beam is potentially unstable on high frequency even with such a damper.

²At this g and Q_0 , the damper exactly suppresses real part of the most dangerous resistive wall harmonic $k = 1$.

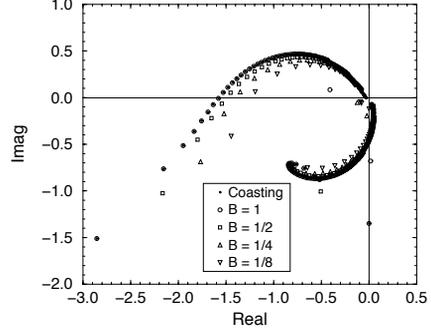


Figure 2: An accelerator with resistive walls and minimal damper: effective impedance multiplied on bunch factor.

UNIFORM BEAM

The mentioned instability can be suppressed by Landau damping owed its origin to spread of incoherent frequency. According to Eq. (8), only chromaticity gives a contribution to the spread what would restricts scopes of the method. However, it is a premature conclusion because slippage factor was neglected in those consideration. Doubtless, the factor is especially important at the analysis of higher modes determining requirements to the damper.

In this part we can restrict ourself by the case $B = 1$ what means a uniform (not a coasting!) beam. Indeed, it follows from previous that, at adiabatic compression of the beam, the effective impedance increases not faster than momentum and frequency spread, i.e. $\propto 1/B$. That is true if the space charge impedance $Z_k^\perp = iZ^{(sc)}$ is taken into account also, adding $iZ^{(sc)}/B$ to any eigenvalue of Eq. (9). Therefore the beam is most unstable at $B = 1$ what case forms the requirements to the damper. Then the matrix $C_{k,k'}$ is:

$$C_{k,k'} = \delta_{k,k'} \int_{-\infty}^{\infty} \frac{\mathcal{F}(p) dp}{\omega + \omega_{rev}(p)Q(p) + k\omega_0\eta(p-p_0)/p_0} \quad (13)$$

$$+ \int_{-\infty}^{\infty} \frac{[\nu + 2(\lambda + \lambda')\Omega][\cos(2\pi\lambda) - \cos(\pi\nu/\Omega)]}{\pi[\lambda + \lambda'][\nu + 2\lambda\Omega][\nu + 2\lambda'\Omega] \sin(\pi\nu/\Omega)} \mathcal{F}(p) dp$$

First part of the formula coincides with similar expression for a coasting beam and corresponds to an approximation of separated harmonics in Eq. (1). Second part owed its origin to the reflection of the particles from the walls is small because the integrand is fast oscillating function at low synchrotron frequency. We can consider it as a small addition to determine limits of applicability of the coasting beam approximation. Then the dispersion equation is:

$$\frac{i r_0 \omega_0 N Z_k^\perp C_{k,k}}{2\pi\gamma Q_0 Z_0} = 1 \quad (14)$$

Further we will consider Gaussian distribution on momentum with dispersion σ_p , and rewrite Eq. (14) in the form:

$$\frac{2\pi i \gamma |\eta \lambda| Q_0 \sigma_p Z_0}{r_0 N p_0 Z_k^\perp} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{x - x_\omega} \times \quad (15)$$

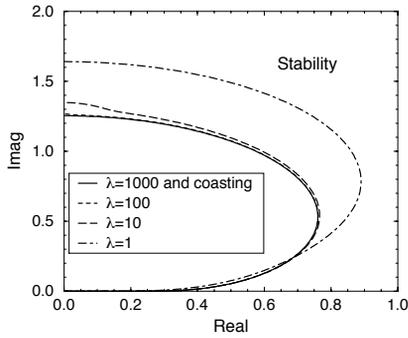


Figure 3: Threshold diagram of a coasting beam and bunched beam at $B = 1$.

$$\left(1 + \frac{[2x - x_\omega][\cos(2\pi\lambda) - \cos(2\pi\lambda x_\omega/x)]}{2\pi\lambda[x - x_\omega] \sin(2\pi\lambda x_\omega/x)}\right) dx$$

where

$$x_\omega = \frac{(\omega + \omega_0 Q_0) p_0}{\omega_0 \sigma_p |\eta \lambda|} \quad (16)$$

The integral can be used to draw a threshold diagram i.e. a mapping of line $\text{Im } x_\omega \rightarrow +0$ on a complex plane. Right-hand part of this symmetric curve is plotted in Fig. 3 at several λ . It is seen that the coasting beam approximation is acceptable at least at $\lambda > 10$. Additionally, one can use the formula:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{x - x_\omega} dx \simeq \frac{x_\omega}{x_\omega^2 - 1} + i \sqrt{\frac{\pi}{2}} e^{-x_\omega^2/2} \quad (17)$$

which provides rather good accuracy at $x_\omega > 2$. Then, for frequent case of dominated space charge impedance, the dispersion equation can be put in a more convenient form:

$$\frac{r_0 p_0 N Z^{(sc)}}{2\pi \gamma \sigma_p Q_0 Z_0 |k - Q_0 \eta + \xi|} = \frac{x_\omega^2 - 1}{x_\omega} \quad (18)$$

$$\frac{\text{Re} Z_k^\perp}{Z^{(sc)}} = \sqrt{\frac{\pi}{2}} \frac{x_\omega^2 - 1}{x_\omega} e^{-x_\omega^2/2} \quad (19)$$

Left-hand part of Eq. (18) is simply the ratio of r.m.s. frequency spread to the space charge tune shift. Right-hand part should be determined by solution of Eq. (19) typically being about 3 – 5. After that, threshold of instability of k -th harmonic can be found from Eq. (18). Minimal of the partial thresholds is the ultimate intensity of the beam.

RECYCLER DAMPER

Now we are applying the results obtained to determine requirements to the Recycler damper. The following parameters are used: $\gamma = 9.526$, $Q_0 = 25.425$, $\eta = -0.00851$, $Z^{(rw)} = 18 \text{ MOhm/m}$ (stainless steel pipe $9.8 \times 4.6 \text{ cm}^2$). Space charge impedance is taken as large as 500 MOhm/m corresponding normalized transverse emittance $1\pi \text{ mm-mrad}$. Longitudinal r.m.s. emittance is 15 eV-s corresponding $\sigma_p = 0.678 \text{ MeV/c}$. Total impedance is calculated by Eq. (12) at $g = 2.64$ what is double minimal gain. Results of are presented in Fig. 4 where threshold of instability is plotted against the chromaticity and the

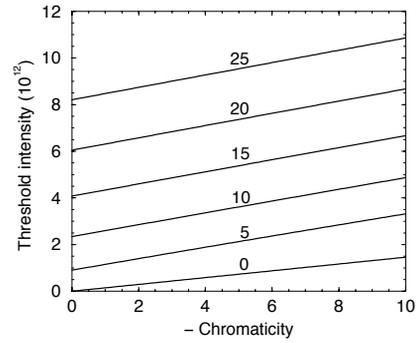


Figure 4: Threshold of instability in the Recycler. Numbers mean the damper bandwidth in MHz (0 – no damper).

damper bandwidth $W = \omega_0/2\pi\alpha$. Without the damper and at nominal chromaticity $\xi = -2$, the threshold is about 3×10^{11} what is confirmed by experiment [2]. At zero chromaticity, the threshold is less by factor ~ 400 . Above the threshold, increment of the lower unstable mode is:

$$\text{Im } \omega \simeq 37 \times 10^{-12} N B^{-1/3} (\text{s}^{-1})$$

what concurs with experiment also. A planed intensity 6×10^{12} can be reached with 18 MHz damper at nominal chromaticity and with 13 MHz one at chromaticity -10 .

CONCLUSION

Modification of general theory of bunched beam transverse instability allows to get simpler equations for a rectangular bunch. Their numerical solution shows that, at given longitudinal emittance, the bunch is most inclined to the instability at $B = 1$. In this case there is no big difference between barrier-bunched and coasting beam, at least for high harmonics and/or at small slippage factor. The techniques developed is applied to explain transverse beam instability in the Fermilab Recycler Ring, and to develop requirements to its damping system. It is shown that the damper bandwidth about 20 MHz is enough to reach planed beam intensity 6×10^{12} ppp at nominal chromaticity.

ACKNOWLEDGMENTS

Author thanks A. Burov, V. Lebedev, and S. Nagaitsev for fruitful discussion.

REFERENCES

- [1] Gerry Jackson, "The Fermilab Recycler Ring Technical Design Report", Fermilab-TM-1991 (1996).
- [2] Martin Hu, James L. Crisp "Fermilab Recycler Damper Requirements and Initial Design". These Proceedings, MPPP011
- [3] F. Sacherer, "Methods for Computing Bunched-Beam Instabilities", CERN-SI-BR-72-5 (1972).
- [4] A. Piwinski, "Impedances in Lossy Elliptical Vacuum Chamber", DESY 94-068 (1994).