ANALYTIC EVALUATION OF THE SERIES OVER AZIMUTHAL HARMONICS AT THE ANALYSIS OF THE STABILITY OF BUNCHED BEAMS COHERENT OSCILLATIONS.

N.V.Mityanina, BudkerINP, Novosibirsk, Russia

Abstract

At the analysis of the stability of coherent motion of multibunch beams including counterrotating beams) one should deal with expressions analogous to the effective impedance - the serieses over harmonics of revolution frequency of the RF structure impedance at the side frequencies to these harmonics, with certain factors depending on the harmonic number, such as the bunch line density spectrum, the phase factor and the factor describing the order of multipole synchrotron oscillations. In this paper, we present the method for analytic summation of these serieses for resonant impedance, which seems not to be made before in the general case including all mentioned factors. Comparison of obtained expressions with formulae used in previous papers shows the limits of validity of simpler approaches. The obtained expressions are used in the computer codes MBIM1 and MBIM2 presented at this conference, which calculate coherent oscillations stability for arbitrary multibunch beams.

THE SERIES TO BE CONSIDERED.

The following series is summarized:

$$S(N,\theta) = \sum_{m=-\infty}^{\infty} m^N e^{im\theta} e^{-m^2\phi_0^2} Z_m^+, \qquad (1)$$

where $Z_m^+ = Z(-i(m\omega_0 + \Omega))$, $Z(-i\omega)$ is the RF cavity impedance, Ω is the frequency of coherent oscillations, ω_0 is the revolution frequency, θ is the angular distance between two bunches of the beam, ϕ_0 is the angular r.m.s. length of a gaussian bunch. The power order N is the positive integer, and for different problems can be both odd and even.

Such series differs from those considered, for example, in [1], first, by the factor $e^{-m^2\phi_0^2}$ which describes the square of a gaussian bunch spectral density, and second, by the fact that the order of the numerator of the algebraic factor of a term of the series is higher than that of the denominator.

Below the formulae for summation of the series (1) will be given for the impedance with the characteristic resistance ρ, resonant frequency $\omega_r = \omega_0 m_r$ and quality factor Q looking like follows

$$Z(s) = \frac{\rho s \omega_r}{(s-s_1)(s-s_2)} = Z^1(s) + Z^2(s), \qquad (2)$$

$$Z^{1,2}(s) = \frac{\rho\omega_r s_{1,2}}{(s_{1,2}-s_{2,1})(s-s_{1,2})},$$

$$s = -i\omega, \quad s_{1,2} = \omega_r(\pm i\sqrt{1-(1/2Q)^2} - (1/2Q)).$$

The impedance (2) contains two resonant summands. It is enough to make summation for the first of them, $Z^1(s)$.

The details of these derivations are given in [4].

SEPARATION OF THE SERIES INTO CONVENIENT PARTS.

For summation, the series (1) should be separated into two parts:

$$S^1(N,\theta) = \sum_{m=-\infty}^{\infty} m^N e^{im\theta} e^{-m^2\phi_0^2} (Z^1)_m^+ =$$

$$= S_P^1(N,\theta) + S_W^1(N,\theta),$$

$$S_P^1(N,\theta) = iR_1 \sum_{m=-\infty}^{\infty} (m^N - m_1^N) \frac{e^{im\theta - m^2\phi_0^2}}{m - m_1}, \qquad (3)$$

$$S_W^1(N,\theta) = iR_1 \sum_{m=-\infty}^{\infty} m_1^N \frac{e^{im\theta - m^2\phi_0^2}}{m - m_1}, \qquad (4)$$

where $m_{1,2} = i(s_{1,2} + i\Omega)/\omega_0$, $R_1 = \dfrac{\rho s_1 m_r}{(s_1 - s_2)}$.

The series $S_W^1(N,\theta)$, containing all the poles and decreasing sufficiently quickly at harmonic numbers $|m| \to \infty$, can be summed up analytically, similarly to the Watson-Sommerfeld transformation (see [2]). The second series, $S_P^1(N,\theta)$, containing no poles, with the help of the Poisson formula (see [4]), will be transformed into the exponentially converging series, from which in most cases it is possible to keep only one or two terms.

SUMMATION OF THE TERM CONTAINING POLES.

The series (4) can be summed up with the help of the Watson - Sommerfeld transformation (see [2]), with some modifications. For that, one should consider auxiliary functions $\phi^{\pm}(z) = f(z)e^{\pm i\pi z}/\sin(\pi z)$, where $f(z) = e^{iz\theta - z^2\phi_0^2}/(z - m_1)$, and their integrals over the

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contours of integration $C^{\pm}$ consisting of the real axis and semicircles of an infinite radius in the upper and lower semiplanes $S^{\pm}$. In [2], the integrals over $S^{\pm}$ had zero limits at infinity, but now they have finite limits, which were calculated in [4]. For $0 \le \theta < 2\pi$, one can get

$$\sum_{m=-\infty}^{\infty} \frac{e^{i\theta m - m^2\phi_0^2}}{m - m_1} = \pi e^{-m_1^2\phi_0^2} e^{i\theta m_1} \times$$
$$\times \left\{ -(ctg(m_1\pi) - i) - i \cdot erfc\left(im_1\phi_0 + \frac{\theta}{2\phi_0} \right) + \right.$$
$$+ \frac{1}{2} e^{2i\pi m_1} erfc\left(im_1\phi_0 + \frac{(2\pi + \theta)}{2\phi_0} \right) -$$
$$\left. - \frac{1}{2} e^{-2i\pi m_1} erfc\left(-im_1\phi_0 + \frac{(2\pi - \theta)}{2\phi_0} \right) \right\}. \quad (5)$$

Usually, ϕ_0 is much less than θ_{min} - the minimal angular distance between bunches, that is why, with expansion of erf ([7], (7.1.6)), at $\phi_0 << \theta_{min}$, one can simplify (5) and get finally

$$S_W^1(N,\theta) = -\pi R_1 m_1^N \left\{ e^{i\theta m_1 - m_1^2\phi_0^2} (ctg(m_1\pi) - \right.$$
$$\left. - i(1 - \delta_{\theta,0})) + \delta_{\theta,0} \frac{2m_1\phi_0}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-2m_1^2\phi_0^2)^k}{(2k+1)!!} \right\}. \quad (6)$$

In the limit $\phi_0 \to 0$, at $N = 0$, the obtained expression (6) coincides with that obtained in [1]. Eq.(6) shows that at $\theta = 0$, the correcting term (in comparison with [1]) obtained now, depends on a ratio of the bunch length to the wavelength of the considered resonant impedance. For the wavelengthes comparable with the bunch length, it is necessary to take into account this correcting term. The sum over k in (6) converges very quickly and usually one can keep only a few terms of it.

SUMMATION OF THE TERM CONTAINING NO POLES.

The series (3) can be written in the form

$$S_P^1(N,\theta) = iR_1 \sum_{k=0}^{N-1} m_1^{N-k-1} C_k, \quad (7)$$

$$C_k = C_k(\theta,\phi_0) = \sum_{m=-\infty}^{\infty} m^k e^{-m^2\phi_0^2} e^{i\theta m}. \quad (8)$$

For calculation of (8) one can apply the Poisson formula (see, for example, [4], eq. (11.1)), for which the sufficient conditions of applicability are fulfilled. According to (2.3.15.9) from [7],

$$C_k = \sum_{n=-\infty}^{\infty} P(\theta + 2\pi n, k), \quad (9)$$

$$P(\theta + 2\pi n, k) = \int_{-\infty}^{\infty} x^k e^{ix(\theta+2\pi n) - x^2\phi_0^2} dx =$$

$$= k! \frac{\sqrt{2\pi}}{(\phi_0\sqrt{2})^{k+1}} e^{-(\theta+2\pi n)^2/4\phi_0^2} \times$$
$$\times \sum_{l=0}^{[k/2]} \frac{\left(-i(\theta+2\pi n)/(\phi_0\sqrt{2})\right)^{k-2l}}{(2l)!!(k-2l)!!}. \quad (10)$$

As $\phi_0 << 2\pi$, at $\theta = 0$, it is possible to drop in (9) the terms with $n \ne 0$, and at $\theta > 0$ - all terms except those for $n = 0,-1$.

The total expression for the sum of the series (1) is calculated by substituting (9) and (10) into (7), adding (6) (or (5)) and summing up terms for $Z^1(s)$ and $Z^2(s)$.

ADDITIONAL NOTES.

1. The same method can be applied to serieses in terms of normal symmetric modes, analogous to (1), but with replacement $\sum_m f(m) \to \sum_p f(pN_b + k)$ (N_b - the number of bunches in symmetric beam, k - the number of the normal mode). The details are given in [4].

2. The same method can be applied in the case of transverse oscillations, for $N = 0$ and for the transverse impedance $Z_t(s) = \frac{i\rho_t\omega_r^2}{(s-s_1)(s-s_2)}$. In this case, first, the size of a bunch can be neglected because the impedance quickly decreases with increasing $|m|$, and second, the summable function sufficiently quickly decreases at $|m| \to \infty$, so the summation formulae similar to those given in [1] can be applied as well as the direct application of the Watson-Sommerfeld formula.

THE CONTRIBUTION OF THE NONRESONANT SUMMAND.

The expression for the sum of series (1) contains two different parts: the resonant summand, containing $(ctg(m_j\pi) - i(1-\delta_{\theta,0}))$ (see (6)) and the nonresonant one (all other terms).

Usually, the papers devoted to summation of similar serieses take into account only the first of these summands, as, for example, in [9].

Using formulae given in this paper, one can show that the contribution of the second (nonresonant) summand at $N > 0$ should not be neglected, if m_r essentially differs from integer numbers and also at low quality factors.

Fig.1 shows the dependences of imaginary parts of resonant and nonresonant summands (divided by $\rho Q m_r$), versus $m_r = \omega_r/\omega_0$, for parameters $\phi_0 = 0.001$, $\Omega/\omega_0 = 0.01$, $\theta = 0$, $N_b = 1$ and for two values of quality factor $Q = 3000$ and

$Q = 30000$ .This simplest example shows that when m_r essentially differ from integer numbers, and also at low quality factors the first summand does not prevail, and the contribution of the nonresonant summand should not be neglected. Note that the major contribution of the nonresonant term is real at even *N* and imaginary at odd *N*.

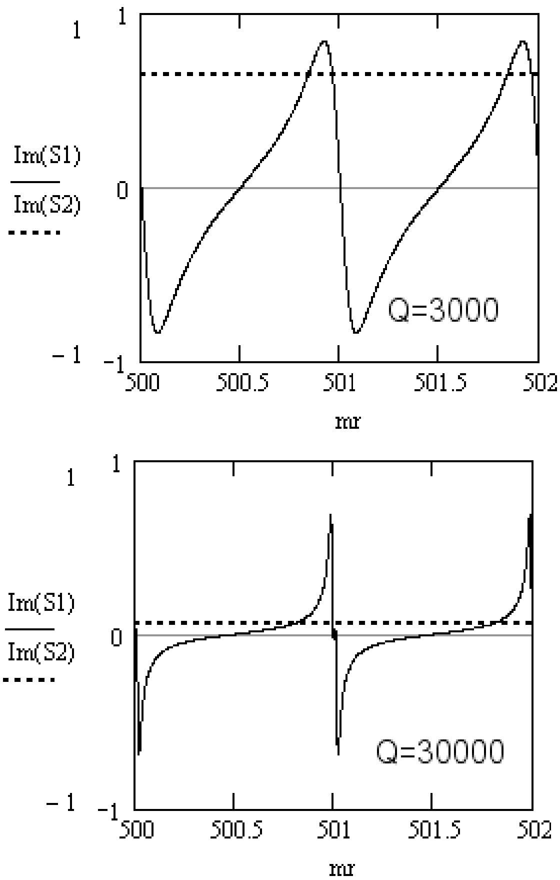

Figure 1: The dependences of imaginary parts of the resonant (S1) and the nonresonant (S2) summands (divided by $\rho Q m_r$), versus $m_r = \omega_r/\omega_0$, for $Q = 3000$ and $Q = 30000$, for $N = 1$.

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