CHARACTERIZATION OF THE CHAOTIC OR REGULAR NATURE OF DYNAMICAL ORBITS: A NEW, FAST METHOD*

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Abstract

A new method of characterization of the regular or chaotic nature of dynamical orbits has been discovered. It takes advantage of both morphological and dynamical properties of orbits, and can be applied to systems of all degrees of freedom. The new technique has been designed to analyze time-independent, time-dependent, and *N*-body systems. It can provide straightforward information about the transition of orbits from regular into chaotic and vice versa, which can happen in time-dependent regimes. Equally important is the distinction it can make between sticky and wildly chaotic epochs during the evolution of chaotic orbits in time-independent regimes. Its most important advantage over the existing methods is that, it characterizes an orbit using information from a very small number of orbital periods. For these reasons the new method is extremely promising to be useful and effective in a broad spectrum of disciplines.

OVERVIEW

The role of chaos in beams has not been completely explored. One of the main reasons is that, there are no available measures of chaos that can characterize reliably and effectively orbits of particles evolving in time-dependent regimes accociated with beams. The existing well-tested measures of chaos are based either on some convergence scheme [1] or on frequency analysis methods [2]. The main representative of the convergent measures is the largest Lyapunov exponent, while frequency analysis methods are almost always based on Fourier analysis.

These measures have three main problems. (a) They need an orbit to evolve in time significantly before they can characterize it reliably. (b) They provide no information about the phenomenon of stickiness [3] (chaotic orbits may stay restrained for a long time in regions which usually surround regular islands in the phase space). (c) The evolution of real beams takes place in a time-dependent regime. Transient chaos [4] (transition of an orbit from chaotic to regular and vice versa) is present in this scenario. The existing measures are able to provide only average-like characterizations in this context. The critical question though remains unanswered: *what parts of an orbit evolving in the time-dependent regime of a beam are regular, and what parts are chaotic?* An answer to that question could provide significant intuition and understanding of the structure of phase space associated with the system, and could give clues about the solution of several problems, e.g. the generation of halos, or the emmitance growth mechanism of a beam.

The new measure [5] addresses all three of these issues. It is based on a pattern recognition scheme and uses all possible information associated with an orbit in order to provide characterization. When tested in time-independent potentials it was able to characterize orbits using only information from 7-15 orbital periods, in most cases. The existing measures usually need at least 25 orbital periods to characterize an orbit, and quite often many more. Moreover, they analyze the orbit as a whole, not as segments: therefore their usefuleness is limited for time-dependent characterizations. The new measure literally dissects the orbit into segments, and then analyzes it. Moreover, it provides information about the parts of chaotic orbits that remain sticky.

If one makes a plot of the extrema of a signal associated with an orbit, it becomes obvious that a regular orbit is characterized by repetitive, smooth patterns [5]. On the other hand, there are no patterns associated with a chaotic orbit in the strict sense that appear in the regular orbits. Still, the signals of the chaotic orbits may be characterized by some loose patterns, which can be used to identify their epochs of stickiness.

In a local level one can look for repetitive patterns in a rather straightforward way. For regular orbits these repetitions in the signals are very easy to identify. A first approach can be a simple definition of *sequential* patterns:

$$|x_k - x_1| \leq \text{irregularity degree}$$
$$|x_{k+1} - x_2| \leq \text{irregularity degree}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$|x_{2k-2} - x_{k-1}| \leq \text{irregularity degree}$$

where the “irregularity degree” is associated with how irregular a pattern is. Regular orbits are characterized by patterns with very small irregularity degrees.

It has to be noted that more sophisticated patterns have also been found and they will be carefully presented in future papers.

NUMERICAL EXPERIMENTS

The application of the new measure in time-independent systems is completely straightforward. The algorithm searches through the signal of an orbit for repetitive patterns. For a regular orbit the whole signal is characterized by repetitive patterns. For chaotic orbits only parts of the signal will be repetitive. One can use this information to construct a clear picture of the phase space.

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The representative example is the so-called Hénon-Heiles potential, which was first used in the context of astronomy in 1963:

$$V(x,y) = \tfrac{1}{2}(x^2 + y^2 + 2x^2 y - \tfrac{2}{3}y^3)$$

This potential is appropriate for our purposes because it admits significant numbers of both regular and chaotic orbits.

The new method was applied to a set of 500 orbits which were evolved for 10 orbital periods (Figure 1) and then for 100 orbital periods ((Figure 2). The plot for the 100 orbital periods is very clear. The four plots show the regular orbits (blue), the sticky parts of the chaotic orbits (green), the wildly chaotic parts of the chaotic orbits (red) and all of them together. Because the new measure can identify chaos or regularity very early in the evolution of an orbit, the plot using the information from only 10 orbital periods, although not perfect, is very decent compared with the one for 100 orbital periods. The basic regions of regularity and chaos are clearly identified. It has to be noted that such plots cannot be produced using the existing measures: many more orbital periods are needed.

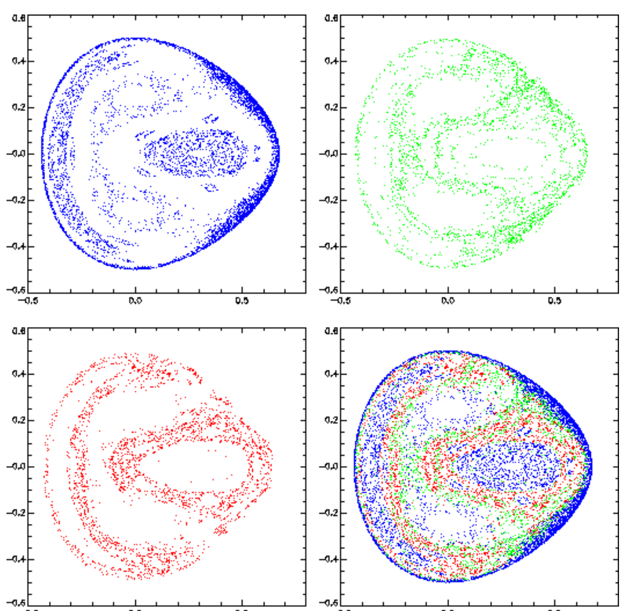

Figure 1. Poincare section of 500 orbits in the Hénon-Heiles potential for evolution of 10 orbital periods. Top left: regular orbits. Top right: sticky epochs of chaotic orbits. Bottom left: wildly chaotic parts of chaotic orbits. Bottom right: all orbits.

Moreover, most measures do not have the capabilities to identify stickiness. The best of them can identify only the very sticky regions around the regular islands. However the new measure was able to identify even relatively difficult sticky regions like the ones associated with the main broken separatrices of the system.

The percentage of success for the new measure was computed by comparing the characterization it provides

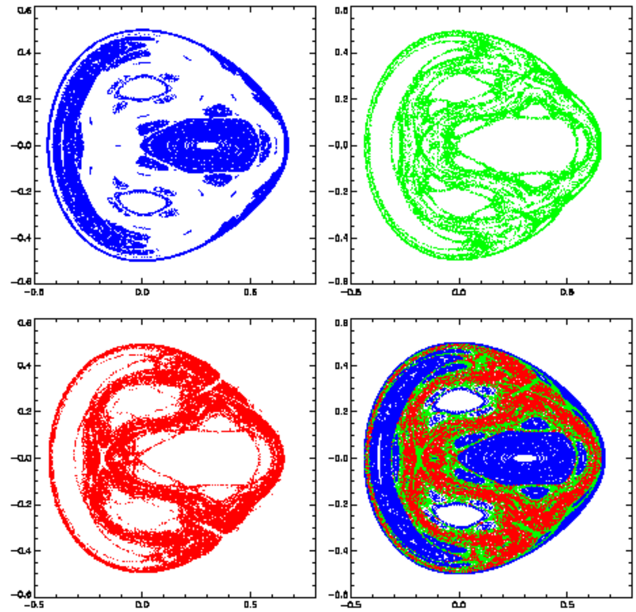

Figure 2. Same as Figure 1 but for evolution equal to 100 orbital periods.

with the characterization by other chaotic measures like Lyapunov exponents or power spectra. The comparisons were made for different evolution times. The new technique is highly successful from early on in the evolution of the orbits. It identifies correctly 90% of the orbits using only the information from 10 orbital periods and it moves to 100% for 100 orbital periods or more. The orbits it fails to identify are the very sticky ones. When the first orbital periods of a chaotic orbit correspond to a sticky epoch, the measure identifies it as regular.

TIME-DEPENDENT POTENTIALS

The established technique for identification of chaos in time-dependent systems is the so-called "frozen-potentials method." This method is based on the concept of freezing the time-dependent potential at successive moments in the evolution of the orbit, then evolving the orbit in that frozen time-independent potential, and characterizing it using any of the existing methods. However this method may fail because it does not necessarily describe correctly what happens in the evolution of the time-dependent regime, but what happens in the corresponding time-independent potentials. The new measure computes how irregular the orbit, which evolved in the time-dependent potential, is. The example potential was a time-dependent version of the dihedral potential:

$$V(x,y,t) = -(x^2 + y^2) + 0.25(x^2 + y^2)^2 - 0.25 m_0 \cos(\omega t) x^2 y^2$$

In order to characterize the orbits in the corresponding frozen potentials the method of complexity was used. This method is based on computing the number of frequencies of the power spectrum of the orbit that consist 90% of the total power. It is obvious from the picture that the frozen-potentials method fails to identify the obvious regularity in the segment 354-374 (Figure 3). The new measure is successful and clearly identifies this segment as regular. Also it is obvious that the two measures agree to a good extent in the segment 100-250.

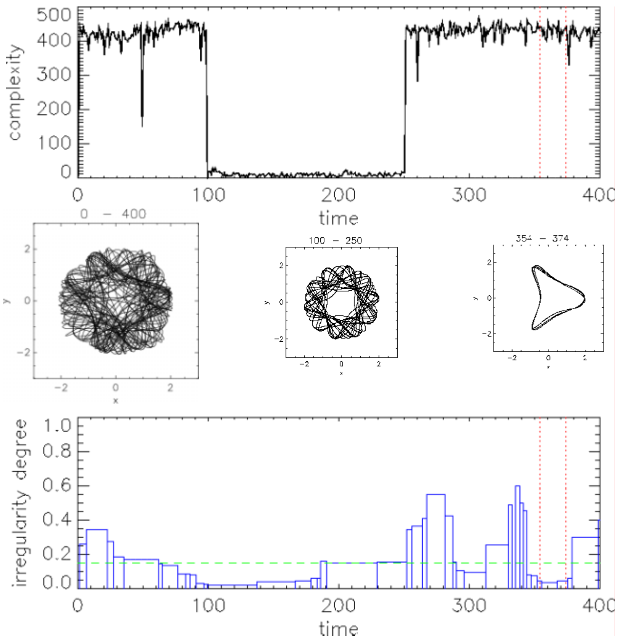

Figure 3: Characterization of an orbit evolved in a time-dependent version of dihedral potential. The top panel shows the complexity (a measure of chaos) of the orbit provided by the frozen-potentials method. The bottom panel shows the characterization of the same orbit using the new technique. In the middle, the left panel shows the whole orbit, the middle panel the segment 100-250, and the right panel the segment 354-374. It is obvious that, although the last segment (354-374) is regular, it is characterized as chaotic by the frozen-potentials method.

The new method was also applied in one orbit evolved in a simulation of an *N*-body system (5-beamlet experiment) [7-8]. It is obvious that the measure identifies parts of the orbit that are not regular. These chaotic points find themselves out of a circle where the regular points are located (Figure 4).

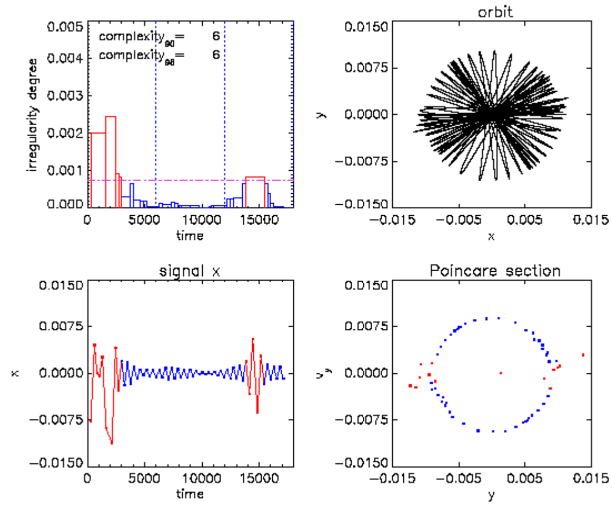

Figure 4: Characterization of an orbit evolved in the *N*-body system of the five-beamlet experiment. Top left panel: the regular (blue) and irregular (red) epochs in the evolution of the orbit. Top right panel: plot of the orbit. Bottom left panel: the signal of the orbit (regular and irregular orbits are identified). Bottom right panel: Poincare section of the orbit. It is obvious that the new technique is successful in identifying points of the orbit that behave irregularly.

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