RESISTIVE-WALL WAKE FOR NON-ULTRARELATIVISTIC BEAMS

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Abstract

We compute the longitudinal and transverse wake fields for velocities smaller than c, and examine under which conditions the non-relativistic terms become important.

1 INTRODUCTION

Severals projects are under construction which aim to produce intense proton or ion beams at energies around 1 GeV, for example, the SNS and J-PARC. One possible performance limitations may arise from the resistive-wall impedance. The conventional treatment of the relativistic wall wake field considers an ultra-relativistic beam; see, *e.g.*, Refs. [1, 2]. Only few papers have attempted to treat the general case. A rare and early example is Ref. [3], but in the ultrarelativistic limit its wake field does not reduce to the conventional expression. In this note we derive the non-ultrarelativistic longitudinal and transverse Green-function wake fields, through first order in the skin depth and second order in $1/\gamma$. We then apply these expressions to four examples.

2 LONGITUDINAL WAKE

We consider a beam pipe of radius b with skin depth $\delta_{\rm skin} = \sqrt{2/(\mu_0\sigma|\omega|)}$ with conductivity σ at angular frequency ω, and assume that the charge density $\lambda \exp(iks - i\omega t)$ travels at the center of the beam pipe with $\omega = vk$ and $v < c$. Following Chao's treatment of the ultrarelativistic case [1], we introduce the new variable $z = (s - vt)$. Then all quantities have the same dependence $\exp(ikz)$ on s and t. The electric and magnetic fields are related to the scalar potential ϕ and the magnetic vector potential $\vec{A}$ via $\vec{E} = -\nabla\phi - \partial\vec{A}/\partial t$ and $\vec{B} = \nabla\times\vec{A}$. Imposing the Lorentz condition $\left(\vec{\nabla}\cdot\vec{A} + (1/c^2)\partial\phi/\partial t + \mu_0\sigma\phi\right) = 0$, the potentials $\vec{A}$ and ϕ fulfill the equations

$$-\triangle\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} + \mu_0\sigma\frac{\partial\vec{A}}{\partial t} = \mu_0(\vec{j} - \sigma\vec{E}) \,. \quad (1)$$

$$-\triangle\phi + \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \mu_0\sigma\frac{\partial\phi}{\partial t} = \frac{\rho}{\epsilon_0} \,. \quad (2)$$

Inside the vacuum chamber $\sigma = 0$. The only nonzero field components are E_s, E_r and B_ϕ because of symmetry. We can thus set $A_\phi = 0$. One further degree of freedom allows us to choose $A_r = 0$ as well.

The Lorentz condition relates the two non-vanishing components as $\phi = (c^2k/\omega)A_s$. The potentials must fulfill

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_s}{\partial r}\right) - k_r^2 A_s = -\mu_0 j_s \,, \quad (3)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) - k_r^2 \phi = -\frac{\rho_0}{\epsilon_0} \,, \quad (4)$$

where $k_r^2 = (k^2 - k_0^2) = k_0^2/(\beta^2\gamma^2) > 0$ with $k_0 = \omega/c$. The right-hand side is zero except for inside the beam. Outside the beam, $A_s = \tilde{A}_s e^{ikz}$ is written as $\tilde{A}_s = pI_0(k_r r) + qK_0(k_r r)$, where the coefficient q is determined by the source current, and p by the surface condition at $r = b$. First we compute q. The source term j_s on the right hand side of (3) is $-\mu_0\lambda v$. The Green function for a point source is $-\ln r/(2\pi)$; the modified Bessel function K_0 expands as $-\ln z$. This yields $q = -\mu_0\lambda v/(2\pi)$. If (r, ϕ, z) are right-handed, the fields are

$$B_\phi = -pk_r I_1(k_r r) + qk_r K_1(k_r r) \,, \quad (5)$$

$$E_s = i\omega\left(1 - \frac{k^2}{k_0^2}\right)(pI_0(k_r r) + qK_0(k_r r)) \,. \quad (6)$$

$$E_r = -\frac{c^2 k k_r}{\omega}(pI_1(k_r r) - qK_1(k_r r)) \,, \quad (7)$$

where we have used $I_0' = I_1$ and $K_0' = -K_1$. The wall is characterized by $\vec{j} = \sigma\vec{E}$ and $\rho = 0$. For $b \gg \delta_{\rm skin}$, inside the wall we have

$$-\frac{\partial^2 E_s}{\partial r^2} \approx -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_s}{\partial r}\right) = (i\mu_0\sigma\omega - k_r^2)E_s \,, \quad (8)$$

with the solution $E_s = E_{s0}\exp(-\lambda r)$, where $\lambda^2 = -(i\mu_0\sigma\omega - k_r^2) = \lambda_0^2 + k_r^2$ and $\lambda_0 \equiv (1 - i\,{\rm sgn}(\omega))/\delta_{\rm skin}$, or

$$\lambda \approx \lambda_0\left(1 + \frac{k_r^2}{2\lambda_0^2}\right) = \lambda_0\left(1 + i\,{\rm sgn}(\omega)\frac{k_r^2\delta_{\rm skin}^2}{4}\right) \,. \quad (9)$$

The second term represents a non-relativistic effect. We will neglect it, since it is of the order $\delta_{\rm skin}^2$.

The relationship between B_ϕ and E_s follows from $\vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t$ and $\vec{\nabla}\cdot\vec{E} = 0$ as $i\omega\tilde{B}_\phi = \tilde{E}_s(1 - i\,{\rm sgn}(\omega))/\delta_{\rm skin}$. Inserting the expressions for B_ϕ and E_s evaluated at $r = b$ and solving for p we get

$$p = -\frac{(1 - i\,{\rm sgn}(\omega))k_r K_0 + \delta_{\rm skin}k_0^2 K_1}{(1 - i\,{\rm sgn}(\omega))k_r I_0 - \delta_{\rm skin}k_0^2 I_1}q \,, \quad (10)$$

where all Bessel functions have the argument $x \equiv k_r b$. The magnetic vector potential $A_s = \tilde{A}_s e^{ikz}$ becomes

$$\tilde{A}_s = -\frac{\mu_0\lambda v}{2\pi}\Big[K_0(k_r r) - \frac{(1 - i\,{\rm sgn}(\omega))K_0(x) + \delta_{\rm skin}k_0^2 K_1(x)}{(1 - i\,{\rm sgn}(\omega))I_0(x) - \delta_{\rm skin}k_0^2 I_1(x)}I_0(k_r r)\Big] \,. \quad (11)$$

Computing from this the longitudinal electric field $E_s = \tilde{E}_s e^{ikz}$ as $\tilde{E}_s = -i\omega k_r^2/k_0^2\tilde{A}_s$, and expanding in k_r and

$\delta_{\rm skin}$, we obtain

$$\tilde{E}_s(k) \approx \frac{\mu_0 \lambda v c k}{2\pi}\left[\frac{(\mathrm{sgn}(k)-i)\delta^{(0)}_{\rm skin}}{2b}\left(1-\frac{1}{4}\frac{k_r^2}{k^2} -\frac{k_r^2(2b^2-r^2)}{4}\right)-i\frac{k_r^2}{k^2}\ln\frac{r}{b}\right] , \quad (12)$$

where $\delta^{(0)}_{\rm skin}$ refers to the skin depth at angular frequency ck, and we have assumed $|k| > k_r$. In the ultrarelativistic limit $k_r \to 0$, the term depending on $\delta^{(0)}_{\rm skin}$ agrees with the result (2.77) in Ref. [1]. If $k_r \neq 0$ the electric field corresponding to the resistive-wall wake depends not only on the longitudinal distance z from the source, but also on the radial position r. The term independent of $\delta_{\rm skin}$ describes the effect of space charge.

The longitudinal impedance per unit length is related to $\tilde{E}_s(k)$ as $Z_{||}(k) = -\tilde{E}_s(k)/j_s$ where $k = \sqrt{k_r^2+\omega^2/c^2}$. From the impedance we compute the longitudinal Green function wake per unit length as $W_0'(z) = -v/(2\pi j_s)\int_{-\infty}^{\infty}\tilde{E}_s(k)e^{ikz}dk$, where, as before, $j_s = \lambda v$. If $W_0'(z) > 0$ the wake field decelerates. Keeping only the res.-wall terms, which depend on $\delta_{\rm skin}$, from (12) we get

$$W_0'(z) \approx -\frac{c^2\mu_0}{4\pi^2}\sqrt{\frac{\pi}{\mu_0\sigma c}}\frac{1}{2b}\beta(\mathrm{sgn}(z)-1) \left(-\frac{\gamma^2-5/4}{(\gamma^2-1)|z|^{3/2}}-\frac{15}{16}\frac{b^2}{(\gamma^2-1)|z|^{7/2}}\right) . \quad (13)$$

In the ultra-relativistic limit this reduces to the familiar

$$W_0'(z) \approx \frac{c^2\mu_0}{4\pi^2}\sqrt{\frac{\pi}{\mu_0\sigma c}}\frac{1}{2b}(\mathrm{sgn}(z)-1)\frac{1}{|z|^{3/2}} . \quad (14)$$

3 TRANSVERSE WAKE

For a transverse wake, every quantity, V, has the dependence $V = \tilde{V}\exp(im\phi+iks-i\omega t) = \tilde{V}\exp(im\phi+ikz)$ on s, ϕ and t. In addition, all field components are nonzero, and we can no longer assume that A_r and A_ϕ are zero. In cylindrical coordinates we find

$$-\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial A_r}{\partial r}\right]+\frac{2im}{r^2}A_\phi+\left[\frac{1+m^2}{r^2}+k_r^2\right]A_r = \mu_0 j_r$$
$$-\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial A_\phi}{\partial r}\right]+\left[\frac{1+m^2}{r^2}+k_r^2\right]A_\phi-\frac{2im}{r^2}A_r = \mu_0 j_\phi$$
$$-\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial A_s}{\partial r}\right]+\left[\frac{m^2}{r^2}+k_r^2\right]A_s = \mu_0 j_s$$
$$-\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial \phi}{\partial r}\right]+\left[\frac{m^2}{r^2}+k_r^2\right]\phi = \frac{\rho_0}{\epsilon_0} .$$

The transverse components are decoupled by introducing $A_+ \equiv A_r + iA_\phi$, $A_- \equiv A_r - iA_\phi$, and $j_\pm \equiv j_r \pm j_\phi$, which yields

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_\pm}{\partial r}\right)-\left(\frac{(1\pm m)^2}{r^2}+k_r^2\right)A_\pm = -\mu_0 j_\pm . \quad (15)$$

The original radial and azimuthal vector components are $A_r = (A_+ + A_-)/2$ and $A_\phi = (A_+ - A_-)/(2i)$.

We only consider the dipole wake, $m = \pm 1$. Outside the beam, the solution for each component is again a superposition of two Bessel functions, *i.e.*, for $m = 1$,

$$\tilde{A}_+^{(1)} = p_+^{(1)} I_2(k_r r) + q_+^{(1)} K_2(k_r r) \quad (16)$$
$$\tilde{A}_-^{(1)} = p_-^{(1)} I_0(k_r r) + q_-^{(1)} K_0(k_r r) \quad (17)$$
$$\tilde{A}_s^{(1)} = p_s^{(1)} I_1(k_r r) + q_s^{(1)} K_1(k_r r) \quad (18)$$
$$\phi = p_0^{(1)} I_1(k_r r) + q_0^{(1)} K_1(k_r r) . \quad (19)$$

and similarly for $m = -1$, for which I_0 (K_0) is exchanged with I_2 (K_2). The coefficients of the Bessel functions for $m = -1$ are called $p_+^{(-1)}$, $q^{(-1)}$, etc., and, by symmetry, they are equal to the corresponding coefficients for $m = 1$, *e.g.*, $p_0^{(-1)} = p_0^{(1)}$ and $p_+^{(-1)} = p_-^{(1)}$. Considering now a perturbation of the form $\cos\phi = (e^{i\phi}+e^{-i\phi})/2$, we obtain

$$A_r = (p_+ I_2 + q_+ K_2 + p_- I_0 + q_- K_0)\cos\phi e^{ikz}$$
$$A_\phi = (p_+ I_2 + q_+ K_2 - p_- I_0 - q_- K_0)\sin\phi e^{ikz}$$
$$A_s = (p_s I_1 + q_s K_1)\cos\phi e^{ikz}$$
$$\phi = (p_0 I_1 + q_0 K_1)\cos\phi e^{ikz} ,$$

where the argument of the Bessel functions is $(k_r r)$, and we have dropped the superindex '(1)' of all coefficients.

The Lorentz condition yields the two equations

$$\frac{k_r}{2}p_+^{(1)}+\frac{k_r}{2}p_-^{(1)}+ikp_s^{(1)}-i\frac{\omega}{c^2}p_0^{(1)} = 0 \quad (20)$$
$$-\frac{k_r}{2}q_+^{(1)}-\frac{k_r}{2}q_-^{(1)}+ikq_s^{(1)}-i\frac{\omega}{c^2}q_0^{(1)} = 0 . \quad (21)$$

To determine these coefficients, we again consider the source terms. As before, $j_\phi = j_r = 0$ and $j_s = v\rho$. But the current source j_s is now displaced by a small transverse distance d from the center of the pipe, so as to generate a dipole moment. The free-space Green function for the dipole current component is $(-\mu_0\lambda dv/r)/(2\pi)$, and the dipole charge source is $(-\lambda d/r)/(2\pi\epsilon_0)$. In the transverse direction $j_+ = j_- = j_r = j_\phi = 0$.

By equating the source terms and their corresponding Green functions with the expansions of K_1, K_0 or K_2 in the expressions for the vector potentials we find that $q_s = -\mu_0\lambda v/(2\pi)\,(k_r d)$, $q_0 = c^2 k/(\omega) q_s$, $q_+ = 0$, $q_- = 0$. Again we invoke the wall boundaries to determine the remaining coefficients. If $|\lambda_0| \gg 1/b$ and $|\lambda_0| \gg k$, we find the same condition as for $m = 0$: $i\omega B_\phi = E_s(1 - i\mathrm{sgn}(\omega))/\delta_{\rm skin} = \lambda_0 E_s$. From Faraday-Maxwell's law we obtain a second boundary condition: $-i\omega B_s \approx \lambda_0 E_\phi$. The longitudinal and azimuthal field components are related to the potentials, $A_r = \tilde{A}_r\cos\phi e^{ikz}$, etc., via

$$B_\phi = \left(ik\tilde{A}_r - \frac{\partial}{\partial r}\tilde{A}_z\right)\cos\phi e^{ikz} \quad (22)$$
$$B_s = \left(\frac{1}{r}\tilde{A}_r + \frac{\partial\tilde{A}_\phi}{\partial r}+\frac{\tilde{A}_\phi}{r}\right)\sin\phi e^{ikz} \quad (23)$$

$$E_\phi = \left(\frac{1}{r}\tilde{\phi} + i\omega\tilde{A}_\phi\right)\sin\phi e^{ikz} \quad (24)$$

$$E_s = \left(-ik\tilde{\phi} + i\omega\tilde{A}_s\right)\cos\phi e^{ikz} \ , \quad (25)$$

so that the boundary conditions at $r = b$ become

$$-\omega k\tilde{A}_r - i\omega\ \partial\tilde{A}_s/\partial r = \lambda_0(-ik\tilde{\phi} + i\omega\tilde{A}_s) \quad (26)$$

$$\lambda_0(\tilde{\phi}/r + i\omega\tilde{A}_\phi) = -i\omega(\tilde{A}_r/r + \partial\tilde{A}_\phi/\partial r + \tilde{A}_\phi/r)\ . \quad (27)$$

The remaining gauge freedom allows for the choice $p_0 = (c^2k/\omega)p_s$. Using this gauge, we can solve the two equations (26) and (27) together with the Lorentz conditions (20) and (21), so as to obtain an expression relating p_s and q_s. Inserting this into the formula for $E_s = \tilde{E}_s\cos\phi e^{ikz}$ and expanding to first order in $\delta_{\rm skin}$ and up to second order in k_r, dropping powers of order higher than 2 in r, we find

$$\begin{aligned}\tilde{E}_s(k) &= -i\omega\frac{k_r^2}{k_0^2}\frac{\mu_0\lambda v}{2\pi}dk_r\left[K_1(k_r r) + \frac{p_s}{q_s}I_1(k_r r)\right]\\ &\approx -ckd\frac{\mu_0\lambda v}{2\pi}\frac{1}{b^3 r}\left[\delta_{\rm skin}^{(0)}\left(r^2 - \frac{1}{4}\frac{k_r^2}{k^2}r^2 - \frac{b^2k_r^2}{4}r^2\right)\right.\\ &\quad\left.(\mathrm{sgn}(\omega) - i) + i\,\mathrm{sgn}(k)b\frac{k_r^2}{k^2}(b^2 - r^2)\right]\ . \quad (28)\end{aligned}$$

The transverse wake is now obtained using the Panofsky-Wenzel theorem $e(E_r + (\vec{v}\times\vec{B})_r) = \int(\partial F_s/\partial r)dz$, which, thanks to $\omega = vk$, is valid as in the relativistic case, and also implies $Z_\perp(k) = Z_{||}(k)/k$. This gives

$$\begin{aligned}\tilde{F}_\perp &\approx \frac{\mu_0 idec\lambda v}{2\pi}\frac{1}{b^3}\left[\delta_{\rm skin}^{(0)}\left(1 - \frac{1}{4}\frac{k_r^2}{k^2} - \frac{b^2k_r^2}{4}\right)\right.\\ &\quad\left.(\mathrm{sgn}(\omega) - i) - i\,\mathrm{sgn}(\omega)b\frac{k_r^2}{k^2}\left(1 + \frac{b^2}{r^2}\right)\right] \quad (29)\end{aligned}$$

The transverse impedance per unit length is $Z_\perp(k) = -i\tilde{F}_\perp/(e\lambda vd)$. In the ultrarelativistic limit, $k_r \to 0$, this agrees with the classical result of [1]. The transverse Green function wake per unit length is $W_1(z) = -1/(2\pi e\lambda d)\int_{-\infty}^{\infty}\tilde{F}_\perp(k)e^{ikz}dk$. If $W_1(z) < 0$, the wake is defocusing. Dropping the space-charge term, from (29) the res.-wall wake function becomes

$$\begin{aligned}W_1(z) &\approx \frac{\sqrt{\pi\mu_0}c^{3/2}}{4\pi^2b^3\sigma^{1/2}}\beta\frac{3b^2 + 4(4\gamma^2 - 5)z^2}{8(\gamma^2 - 1)|z|^{5/2}}\\ &\quad(\mathrm{sgn}(z) - 1)\ . \quad (30)\end{aligned}$$

In the ultra-relativistic limit this reduces to

$$W_1(z) \approx \frac{\sqrt{\pi\mu_0}c^{3/2}}{2\pi^2b^3\sigma^{1/2}}\frac{1}{|z|^{1/2}}(\mathrm{sgn}(z) - 1)\ . \quad (31)$$

4 APPLICATIONS

Typical parameters of several low-energy proton or ion accelerators are listed in Table 1. For each case we consider a stainless chamber with $\sigma = 1.4 \times 10^6\ \Omega^{-1}\ \mathrm{m}^{-1}$. The longitudinal wake function at the chamber wall are shown in Fig. 1, the transverse in Fig. 2. The differences between the ultra-relativistic limit, (14) and (31), and the more accurate formulae, (13) and (30), are significant for $z > -b/\sqrt{10\gamma^2 - 7}$ or $\gamma < 3$. The energy decreases from SNS, over J-PARC and PS booster to an ECR source. The latter also illustrates the effect of a smaller pipe radius b.

Table 1: Example Parameters

	SNS	J-PARC	PS booster	ECR
γ	2.1	1.4	1.05	1.003
σ_z	25 m	12 m	26 m	100 m
σ_r	2 cm	2 cm	3 mm	4 mm
b	8 cm	12.5 cm	30 cm	3 cm
QN_b	1.5×10^{14}	4×10^{13}	1.2×10^{12}	2×10^{13}

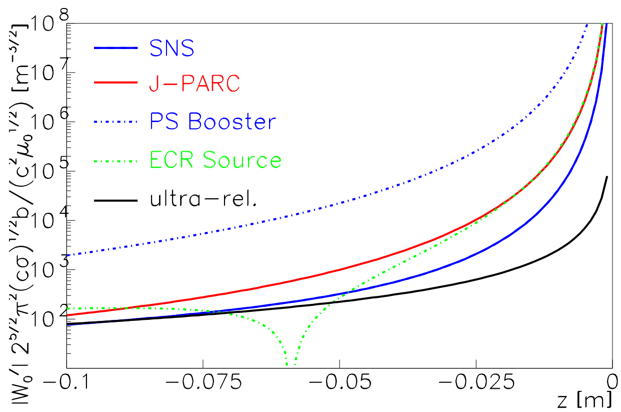

Figure 1: Longitudinal wake $|W_0'|2^{5/2}\pi^2b\sqrt{\sigma c}/(c^2\mu_0)$ at $r = b$ vs. distance z in m, according to (13) [colored] and in the ultra-relativistic limit (14) [black solid].

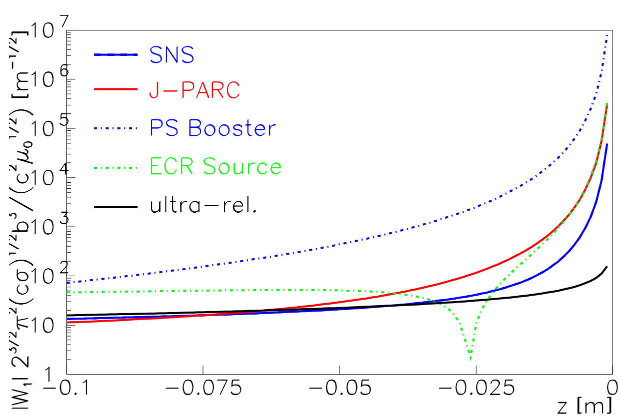

Figure 2: Transverse r. w. wake $|W_1|2^{3/2}\pi^2b^3\sqrt{\sigma c}/(c^2\mu_0)$ vs. distance z in m, according to (30) [colored] and in the ultra-relativistic limit (31) [black solid].

5 REFERENCES

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