# LINEAR DESIGN OF COMBINED-FUNCTION IONIZATION COOLING LATTICES

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*Abstract*

Ionization cooling lattices simultaneously require small beta-functions at the absorber and large energy acceptances to be effective. Simultaneously achieving these goals as well as having a good dynamic aperture requires that the lattice be relatively compact. If one wishes to avoid solenoids, one choice for creating such a lattice is to use combined-function magnets. These magnets can simultaneously focus in both planes, allowing one to achieve a low beta in both planes with a minimum number of magnets. In this paper we explore the design of lattices which contain only combined-function bending magnets using a thin-lens approximation, showing how to optimally achieve the requirements for muon cooling.

## INTRODUCTION

Ionization cooling is needed in most scenarios for muon-based accelerators such as neutrino factories and muon colliders. Cooling rings have been proposed as a cost-effective means for cooling in all six phase space dimensions [1, 2, 3]. Cooling rings have traditionally used solenoids, but it is not absolutely necessary to do so.

Since one must bend and focus in the cooling ring, it is logical to use combined-function bends in a non-solenoid cooling ring. For the simple designs here, symmetry will dictate that one chooses the magnet gradients to focus equally in both planes. Thus, all analyses here will be done in a single plane.

For a combined-function bend which focuses equally in both planes, the integrated focusing strength $F$ is

$$F = \frac{Bc\theta}{2pc/q}, \tag{1}$$

where $B$ is the dipole component of the field, $\theta$ is the bending angle, $p$ is the momentum, $q$ is the particle's charge, and $c$ is the speed of light. Note the inverse dependence of $F$ on momentum.

There are two parameters that are of importance in a cooling lattice: the energy bandwidth (should be large) and the beta function at the absorber (should be small). They compete with each other: when the energy bandwidth increases, so does the maximum beta function. The energy bandwidth is described by the half-width of the relative momentum spread $\Delta$:

$$\Delta = (p_{\text{max}} - p_{\text{min}})/(p_{\text{max}} + p_{\text{min}}), \tag{2}$$

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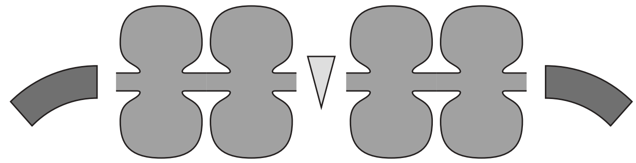

Figure 1: Layout with one bend per cell.

where $p_{\text{min}}$ and $p_{\text{max}}$ are the minimum and maximum momentum respectively. It is also useful to define a reference momentum to be

$$p_{\text{ref}} = (p_{\text{max}} + p_{\text{min}})/2 \tag{3}$$

In these calculations, the change in the closed orbit is ignored. As the closed orbit changes, the focusing strength in the combined function bend will change. In particular, a bending magnet that focuses equally in both planes at the reference energy does not do so off-energy.

## ONE BEND PER CELL

If there is only one bend per cell, as shown in Fig. 1, and the cell length is $L$, then the beta function at the absorber is

$$L\sqrt{\frac{1 - LF/4}{LF}}. \tag{4}$$

This lattice is unstable when $LF > 4$; thus, there is a momentum threshold below which the lattice is unstable. The lattice is stable for arbitrarily large momentum. The maximum $\beta$-function at the absorber occurs at the largest momentum within the passband. Thus,

$$\beta_{\text{max}} = 2L\sqrt{\frac{2\Delta}{1-\Delta}} \qquad \Delta = \frac{\beta_{\text{max}}^2}{8L^2 + \beta_{\text{max}}^2}. \tag{5}$$

The required magnetic field is

$$B\theta = \frac{2}{Lc}\frac{p_{\text{ref}}c}{q}(1-\Delta) \tag{6}$$

$$= \frac{2}{c}\frac{p_{\text{ref}}c}{q}\frac{8L}{8L^2 + \beta_{\text{max}}^2} = \frac{2}{Lc}\frac{p_{\text{min}}c}{q}. \tag{7}$$

Note that if $\beta_{\text{max}}$ is much less than $L$, then the energy acceptance is very small. Thus, this is probably not an optimal configuration for a cooling ring.

## TWO BENDS PER CELL

Now, consider a lattice with two bends per cell, as shown in Fig. 2. Symmetry considerations dictate that the two

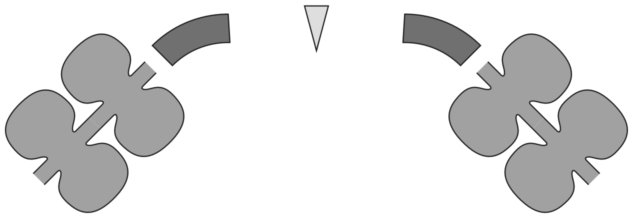

Figure 2: Layout with two bends per cell.

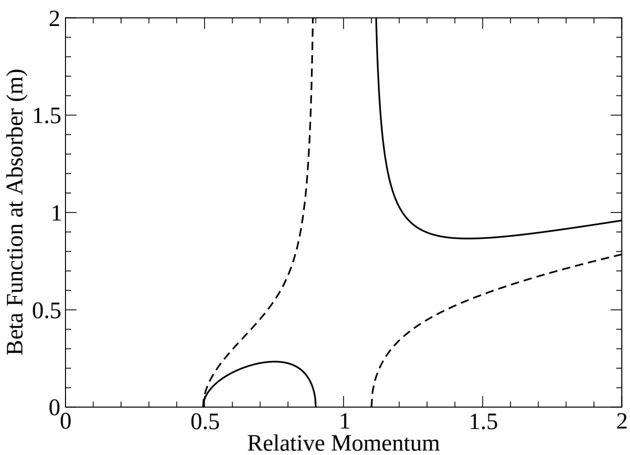


Figure 3: Beta as a function of momentum. The solid line is for $L_A = 0.9$ m and $L_R = 1.1$ m, and the dashed line is for $L_A = 1.1$ m and $L_R = 0.9$ m.

bends be identical. The beta function at the absorber is

$$\frac{L_A}{2}\sqrt{\frac{\left(\frac{2}{L_A}-F\right)\left(\frac{2}{L_A}+\frac{2}{L_R}-F\right)}{\left(\frac{2}{L_R}-F\right)F}}, \tag{8}$$

where $L_A$ is the length of the drift containing the absorber, and $L_R$ is the length of the other drift (presumably containing RF).

As a function of momentum, the $\beta$-function has two zeros, when $F = 2/L_A$ and when $F = 2/L_A + 2/L_R$. There are two poles, when $F = 0$ and $F = 2/L_R$. There are thus two passbands: from the $F = 2/L_A + 2/L_R$ to the larger of $2/L_A$ and $2/L_R$, and from the smaller of $2/L_A$ and $2/L_R$ and 0. These passbands are given in terms of $F$, but they are equivalent to passbands that are determined by the momentum. One passband is unbounded in momentum, and the other is a bounded region of momentum. The bounded region can either have zeros at both ends or a zero at one end and a pole on the other, as shown in Fig. 3. If the bounded region has an infinite $\beta$-function at one end, then the unbounded region will have a zero in the beta function $\beta$-function at the low momentum end, and will go to infinity as the momentum goes to infinity. Thus, the $\beta$-function gets large in either region. However, if the $\beta$-function is zero on both sides of the bounded region (it cannot be zero on both sides of the unbounded region), it will have a maximum value in that region. As can be seen in Fig. 3, one can achieve a lower maximum beta by running in this bounded region with zeros of the beta function on both ends of the region. The peak $\beta$-function can be written in terms of $\Delta$ as

$$\begin{aligned}\frac{L_R-\sqrt{L_R^2-L_A^2}}{2} &= \frac{L_A}{2}\left(\frac{1-\Delta}{2\Delta}-\frac{\sqrt{1-2\Delta-3\Delta^2}}{2\Delta}\right)\\ &= \frac{L_R}{2}\left(1-\frac{\sqrt{1-2\Delta-3\Delta^2}}{1-\Delta}\right).\end{aligned} \tag{9}$$

Inverting this equation gives

$$\Delta = \frac{\sqrt{\beta_{\text{max}}(L_R-\beta_{\text{max}})}}{L_R+\sqrt{\beta_{\text{max}}(L_R-\beta_{\text{max}})}}. \tag{10}$$

Note that the maximum $\Delta$ is 1/3 (giving an energy acceptance of a factor of 2), and occurs when $\beta_{\text{max}} = L_R/2$. The relationship between the lengths is

$$L_A = L_R\frac{2\Delta}{1-\Delta} = 2\sqrt{\beta_{\text{max}}(L_R-\beta_{\text{max}})}, \tag{11}$$

or inverting,

$$\Delta = \frac{L_A}{2L_R+L_A}. \tag{12}$$

Finally, the field strength required is given by

$$B\theta = \frac{2}{cL_R}\frac{p_{\text{ref}}c}{q}\frac{1-\Delta^2}{\Delta} \tag{13}$$

$$= \frac{8}{c}\frac{p_{\text{ref}}c}{q}\frac{L_R+L_A}{L_A(2L_R+L_A)}. \tag{14}$$

As one reduces $\beta_{\text{max}}$ relative to $L_R$, the energy acceptance decreases, and the field required in the bending magnets increases.

## EXAMPLE

Let's say that a 1 m drift is needed for the RF, and the maximum desired beta function at the absorber is 0.25 m. The reference momentum for the lattice is 200 MeV/$c$.

With only one bend per cell, the energy acceptance of this lattice is less than 1%. This is unacceptable for a cooling lattice. If there are two bends per cell, the energy acceptance is $\pm 30\%$. The length of the absorber straight is 0.87 m. $B\theta$ is about 4 T; if we have 8 bends in the ring, this corresponds to bending fields of 5.1 T, and correspondingly lower if one uses fewer bends. The length of this bend is about 10 cm. Using 4 bends gives a 2.6 T field with 41 cm bend lengths.

The required magnet apertures are difficult to determine, since the beta functions go to infinity as the ends of the momentum passband; an energy cutoff must be introduced because of that, or the ellipsoidal beam distribution must be taken into account.

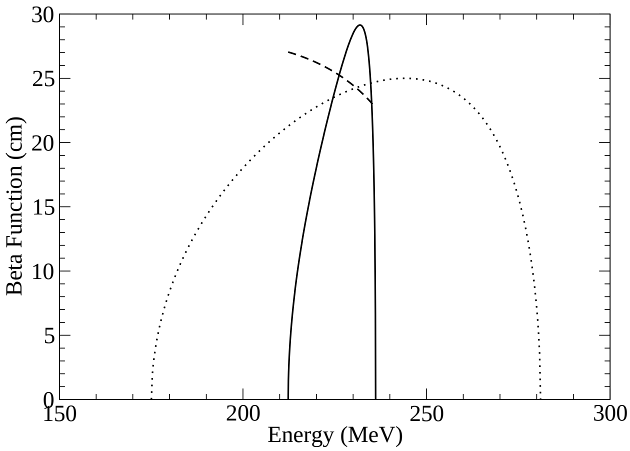


Figure 4: Beta function at the absorber as a function of energy for combined-function magnets with linear midplane field dependence. The dotted line is the thin-lens model, the solid line is $\beta_x$, and the dashed line is $\beta_y$.

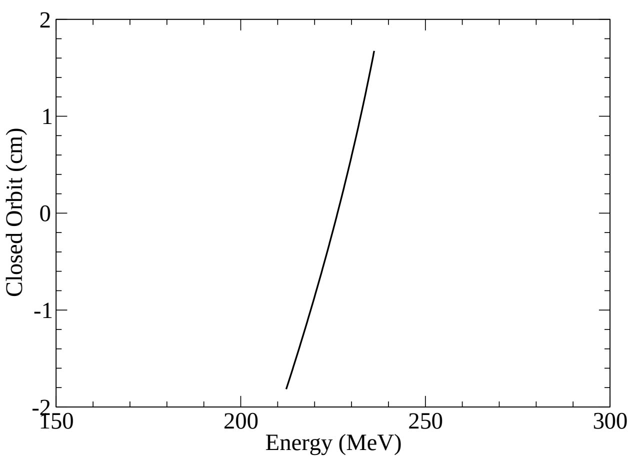


Figure 5: Closed orbit as a function of energy at the absorber side of the magnet for combined-function magnets with linear midplane field dependence.

### *Thick Magnets*

Now, instead of using a thin-lens model, use thick combined-function bends. If we use a bend whose field is linear in the midplane, we get the beta functions shown in Fig. 4. Note the substantially reduced energy acceptance from the linear model. The reason for this reduced energy acceptance is the closed orbit variation with energy (see Fig. 5). When the closed orbit is at a larger radius, the vertical focusing is increased since the length of the orbit in the magnet is longer, but the horizontal focusing is decreased since the geometric contribution to focusing is reduced due to the larger radius of curvature.

One can try to correct the chromaticity by adding a sextupole component to the magnets. The optimal way to do this is to use a magnet which maintains equal focusing in both planes at all positions in the magnet: a magnet whose

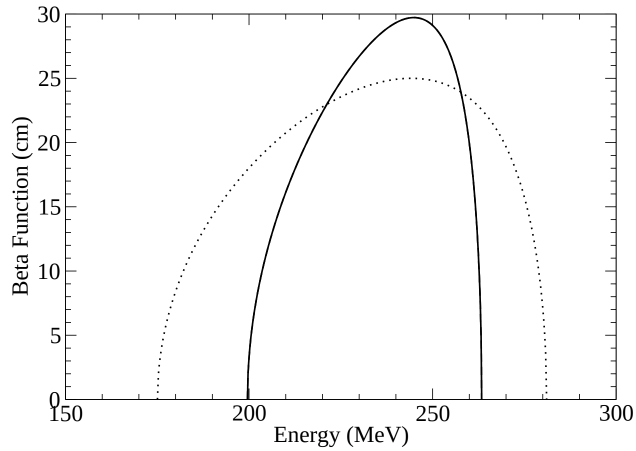


Figure 6: Beta function at the absorber as a function of energy for power law midplane field dependence. The dotted line is the thin-lens model; the solid line is for the actual magnet (both planes).

midplane field is

$$B(x) = B_0\,(1 + x/\rho)^{-1/2}\,. \qquad (15)$$

Figure 6 shows the beta functions at the absorber in this case. The energy acceptance has been greatly improved from Fig. 4, but is still not as large as the linear model suggests. Nor should one expect improvement from here: since the beta functions are identical in the two planes, changing the sextupole component would likely make the linear resonances in one plane closer together, while moving them further apart in the other plane.

## CONCLUSIONS

A thin-lens model can be used to approximately design and predict the qualitative behavior of a simple cooling ring using only combined-function bending magnets. Probably the most important use of this is to predict the parametric dependence of and interrelationship between performance parameters for such a cooling ring.

A simple design with aggressive lattice parameters seems to be a bit unrealistic: the magnets are very short, and the ring circumference is very low as well. With less ambitious lattice parameters (a relaxed beta function, for instance), the lattice may end up being more realistic.

This paper doesn't look at the dynamic aperture of these lattices: the use of highly nonlinear magnets may have a negative impact on that.

## REFERENCES


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