EVOLVING BUNCH AND RETARDATION IN THE IMPEDANCE FORMALISM*

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Abstract

The usual expression for the longitudinal wake field in terms of the impedance is exact only for the model in which the source of the field is a rigid bunch. To account for a deforming bunch one has to invoke the *complete impedance*, a function of both wave number and frequency. A computation of the corresponding wake field would be expensive, since it would involve integrals over frequency and time in addition to the usual sum over wave number. We treat the problem of approximating this field in an example of current interest, the case of coherent synchrotron radiation (CSR) in the presence of shielding by the vacuum chamber.

Consider a rigid bunch moving on a circular trajectory of radius R. A test particle feels a voltage

$$V(\theta,t)=\omega_0 Q\sum_{n=-\infty}^{\infty} e^{in(\theta-\omega_0 t)}Z(n)\lambda_n\ , \qquad (1)$$

where $\theta-\omega_0 t$ is the azimuthal angle between the test particle and the reference particle, the latter having revolution frequency $\omega_0=\beta_0 c/R$. The impedance at azimuthal mode number n (wave number n/R) is $Z(n)$. The mode amplitude λ_n is the Fourier transform of the line density $\lambda(\theta)$ in the bunch rest frame. With $\int\lambda(\theta)d\theta=1$ the total charge is Q. For the case of a deforming bunch one's first inclination is merely to replace λ_n in (1) by $\lambda_n(t)$. The resulting formula (or its equivalent statement in terms of a wake potential) has been used in dynamical calculations based on the Vlasov-Fokker-Planck equation [1, 2, 3, 4], and in earlier macroparticle simulations. In such work λ_n (or $\lambda(\theta)$) is updated at each time step according to the values of external and coherent forces at the previous step. Although the calculations seem successful in many respects, the simple replacement $\lambda_n\ \rightarrow\ \lambda_n(t)$ is a first approximation of uncertain accuracy, especially in an unstable regime of rapid bunch evolution.

Our object here is to derive this first approximation and systematic corrections. We do so in an analytically solvable model, the case of particles on circular trajectories between two infinite parallel plates, perfectly conducting. The plates represent the vacuum chamber, which suppresses CSR at wavelengths greater than a certain "shielding cutoff". This model has considerable utility in spite of its simplified view of a real system; it led to interesting results on instabilities induced by CSR in the work of Refs. [2, 3].

The field equations are solved in cylindrical coordinates (r,θ,y), with y-axis perpendicular to the plates and origin midway between the plates of separation h. We allow an arbitrary but fixed distribution of charge in the y-direction, with density $H(y)$, $\int H(y)dy=1$. The full charge density has the form $\rho(\theta,t)=Q\lambda(\theta-\omega_0 t,t)H(y)\delta(r-R)/R$. We make a Laplace transform of the Maxwell equations and the charge/current density in time, assuming that the charge and the fields are zero before $t=0$. We also make Fourier transforms in θ and y, the Fourier series in y chosen to satisfy the boundary conditions of fields on the plates. Then the transformed field equations can be solved in terms of Bessel functions. The Fourier/Laplace transform of the longitudinal electric field (averaged over the vertical distribution $H(y)$) will be denoted as $\hat{E}(n,\omega)$. Here $\omega=u+iv$, $v>0$ is a complex frequency, while the Laplace transform variable conjugate to time is $s=-i\omega$. By linearity of the field equations, $-2\pi R\hat{E}$ is proportional to the corresponding transform of the current, with a proportionality constant $Z(n,\omega)$ called the *complete impedance*: $-2\pi R\hat{E}(n,\omega)=Z(n,\omega)\hat{I}(n,\omega)$.

The transform of the current is

$$\hat{I}(n,\omega)=\frac{Q\omega_0}{2\pi}\int_0^\infty dt e^{i(\omega-n\omega_0)t}\lambda_n(t)\ , \qquad (2)$$

$$\lambda_n(t)=\frac{1}{2\pi}\int_0^{2\pi}d\theta e^{-in\theta}\lambda(\theta,t)\ . \qquad (3)$$

Compare the case of a rigid bunch existing for all time, for which

$$\hat{I}(n,\omega)=Q\omega_0\delta(\omega-n\omega_0)\ . \qquad (4)$$

The impedance has the form

$$Z(n,\omega)=\frac{Z_0(\pi R)^2}{\beta_0 h}\sum_{p=1}^{\infty}\Lambda_p\Big[\frac{\omega\beta_0}{c}J'_{|n|}(\gamma_p R)H^{(1)\prime}_{|n|}(\gamma_p R)$$
$$+\Big(\frac{\alpha_p}{\gamma_p}\Big)^2\frac{n}{R}J_{|n|}(\gamma_p R)H^{(1)}_{|n|}(\gamma_p R)\Big]\ . \qquad (5)$$

Here $H_n^{(1)}=J_n+iY_n$, where J_n and Y_n are Bessel functions of the first and second kinds, $Z_0=120\pi\ \Omega$ in m.k.s. units, and $\alpha_p=\pi p/h$, $\gamma_p^2=(\omega/c)^2-\alpha_p^2$. The sum on p corresponds to modes in the Fourier expansion with respect to y. The factor Λ_p depends on the vertical distribution $H(y)$, and is zero for even p if H is even. For a Gaussian distribution with r.m.s. width $\sigma_y\ll h$, and the y-average to define $\hat{E}$ taken over $[-\sigma_y\ ,\ \sigma_y]$, we have $\Lambda_p=2\sin(x)e^{-x^2/2}/x\ ,\ x=\alpha_p\sigma_y$.

* Work supported in part by Department of Energy contract DE-FG03-99ER41104

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The impedance at fixed integer n is defined as an analytic function of ω in the upper half-plane by first defining $\gamma_p(\omega) = ((\omega/c)^2 - \alpha_p^2)^{1/2}$ as an analytic function. We take $\gamma_p(u)$ to be positive for $u > \alpha_p c$, then define $\gamma_p(\omega)$ by analytic continuation to $\mathrm{Im}\ \omega \geq 0$; it follows that $\gamma_p(-u) = -\gamma_p(u)\ ,\ u > \alpha_p c$. This specification makes $Z(n,\omega)$ analytic and bounded for $\mathrm{Im}\ \omega \geq v > 0$. The boundedness follows from integral representations and asymptotic formulas [3]. On the real axis Z is not bounded, having poles at the wave guide cutoffs, $\omega = \pm\alpha_p c$. These points are frequency thresholds for the advent of propagating waves with transverse mode number p. In (5) the poles are from γ_p^{-2} in the coefficient of $J_{|n|}Y_{|n|}$ and from $J'_{|n|}Y'_{|n|}$. The poles alone make the following contribution to the impedance, for $|n| > 0$:

$$Z_*(n,\omega) = i\frac{Z_0\pi c}{2\beta_0 h}\sum_p \Lambda_p \times\left[\frac{|n|\beta_0 - \mathrm{sgn}(n)\alpha_p R}{\omega - \alpha_p c} + \frac{|n|\beta_0 + \mathrm{sgn}(n)\alpha_p R}{\omega + \alpha_p c}\right] , \tag{6}$$

where $\mathrm{sgn}(n)$ is the sign of n. There is no pole for $n = 0$. The poles do not show up as infinities or even sharp peaks in $Z(n) = Z(n, n\omega_0)$, since $Z_*(n, n\omega_0) = iZ_0(\pi R/\beta_0 h)\sum_p \Lambda_p$ is bounded and independent of n.

From (2) and the definition of Z we get the wake voltage V by the inverse Fourier/Laplace transform as

$$\begin{aligned} &V(\theta,t) = -2\pi R\mathcal{E}(\theta,t) = \\ &\omega_0 Q\sum_n e^{in\theta}\int_{\mathrm{Im}\ \omega=v} d\omega e^{-i\omega t}Z(n,\omega) \\ &\times\frac{1}{2\pi}\int_0^\infty dt' e^{i(\omega-n\omega_0)t'}\lambda_n(t') \ . \end{aligned} \tag{7}$$

If we substitute the rigid bunch current instead of (2) in (7), we get (1) with the identification $Z(n) = Z(n, n\omega_0)$. We cannot be sure that the ω-integral in (7) exists without some assumption on λ_n, since $Z(n, u+iv)$ is bounded but nonvanishing as $|u| \to \infty$; see [3]. We assume that $\lambda_n(t) \in C^{(2)}(-\infty,\infty)$, i.e., it has a continuous second derivative on the real line. Then since $\lambda_n(t) = 0\ ,\ t < 0$, it follows that $\lambda_n^{(k)}(0) = 0\ ,\ k = 0,1,2$. This allows two partial integrations with vanishing boundary terms, so that the t'-integral takes the form

$$-\frac{1}{(\omega - n\omega_0)^2}\left(\int_0^t + \int_t^\infty\right)dt' e^{i(\omega-n\omega_0)t'}\lambda''_n(t') , \tag{8}$$

which is $\mathcal{O}(u^{-2})\ ,\ u \to \infty$. Consequently, the ω-integral converges absolutely. Moreover, the second term in (8) contributes nothing to (7), since it is analytic for $\mathrm{Im}\ \omega = v > 0$ and is less in magnitude than $M\exp(-vt)/|\omega|^2$ for some constant M. Since $Z(n,\omega)$ is analytic and bounded for $\mathrm{Im}\ \omega > 0$, we can replace the ω-integral by an integral over the semi-circle at infinity, which is zero, thanks to decay of the integrand as $|\omega|^{-2}$. Thus causality is satisfied, since future values of the charge density do not enter:

$$V(\theta,t) = -\omega_0 Q\sum_n e^{in\theta}\int_{\mathrm{Im}\ \omega=v} d\omega e^{-i\omega t}Z(n,\omega) \times\frac{1}{(\omega-n\omega_0)^2}\frac{1}{2\pi}\int_0^t dt' e^{i(\omega-n\omega_0)t'}\lambda''_n(t') \ . \tag{9}$$

One could attempt a calculation of V by direct numerical evaluation of the two integrals in (9), but that would be expensive and would involve many insignificant contributions. Instead we note that the t'-integral is expected to be concentrated (for small v) near $u = n\omega_0$. Outside such a neighborhood the integral is small by virtue of oscillations. Moreover, the second order pole at $\omega = n\omega_0$ also tends to concentrate the ω integral (i.e, u-integral) near $u = n\omega_0$. Consequently, it makes sense as a first approximation to replace $Z(n, n\omega)$ by $Z(n,\omega_0)$, and that allows us to compute the ω-integral by closing the contour in the lower half-plane. The factor

$$-\frac{1}{2\pi}e^{-i\omega t}\int_0^t e^{i(\omega-n\omega_0)t'}\lambda_n(t')dt' \tag{10}$$

is an entire function of ω, bounded in the lower half-plane, and the residue of the second order pole is just the derivative of (10), evaluated at $n\omega_0$. An integration by parts using $\lambda_n(0) = \lambda'_n(0) = 0$ then gives the expected lowest approximation, namely (1) with $\lambda_n \to \lambda_n(t)$, and $Z(n) = Z(n, n\omega_0)$.

For the next approximation one might think of expanding $Z(n,\omega)$ in (9) in a Taylor series about $\omega = n\omega_0$. This cannot succeed for $n\omega_0$ close to $\pm\alpha_p c$, because of the poles of (6). We can, however, write $Z(n,\omega) = \tilde{Z}(n,\omega) + Z_*(n,\omega)$ and compute the contribution of the pole term Z_* to the ω-integral by the method of residues. The smooth remainder $\tilde{Z}$ can later be expanded in a Taylor series about $n\omega_0$. For the integral of the pole term it is best to first undo the two partial integrations in t', so that the ω-integral of (9) has just first order poles at $\pm\alpha_p c$. The boundary terms vanish as is seen by closing their integrals by a circle at infinity in the upper half-plane. The integral over Z_* is then found by closing the contour in the lower half plane, and the result is

$$-\frac{Z_0\pi R e^{-in\omega_0 t}}{2\beta_0 h}\sum_p \Lambda_p\int_0^t \lambda_n(t')dt' \times\left[A(p,n)e^{iA(p,n)(t'-t)} + B(p,n)e^{iB(p,n)(t'-t)}\right] , \tag{11}$$

where $A(p,n) = \alpha_p c - n\omega_0\ ,\ B(p,n) = -\alpha_p c - n\omega_0$.

Having accounted exactly for the poles, we account approximately for the remainder $\tilde{Z}$ by a Taylor expansion, $\tilde{Z}(n,\omega) = \tilde{Z}(n,n\omega_0) + \partial\tilde{Z}(n,n\omega_0)/\partial\omega\ (\omega - n\omega_0) + \cdots$. To evaluate the contribution of the k-th order term of the expansion to (9), we have to assume $\lambda_n(t) \in C^{(k+2)}(-\infty,\infty)$. Taking the expansion just to the first order, we suppose that λ_n has a continuous third derivative,

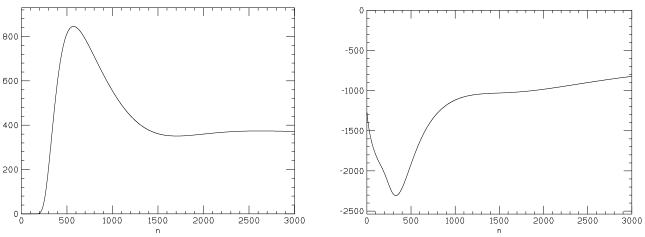

Figure 1: Real and imaginary parts of $\omega_0 \partial \tilde{Z}(n, n\omega_0)/\partial\omega$ in ohms, plotted versus n

and do an additional integration by parts on t' to get a factor $(\omega - n\omega_0)^{-3}$, thus a behavior $\mathcal{O}(|\omega|^{-2})$ in the lower half-plane for the first order Taylor term, enough to close the contour in the lower half-plane. The zeroth order Taylor term contains a contribution from $-Z_*(n, n\omega_0)$ which is finite but alarmingly large. This caused a headache until we realized that it is exactly cancelled by a part of (11), namely the boundary term in an partial integration in which the integral over λ_n is replaced by an integral on λ_n'. Invoking this cancellation, we state our proposal for the wake voltage with main corrections to the lowest approximation:

$$V(\theta, t) = 2\omega_0 Q \mathrm{Re} \sum_{n=1}^{\infty} e^{in(\theta - \omega_0 t)} \Big[Z(n, n\omega_0)\lambda_n(t) + i\frac{\partial \tilde{Z}}{\partial\omega}(n, n\omega_0)\lambda_n'(t) - i\frac{Z_0 \pi R}{2\beta_0 h} \sum_p \Lambda_p \times \int_0^t dt' \lambda_n'(t') \Big(e^{iA(p,n)(t'-t)} + e^{iB(p,n)(t'-t)} \Big) \Big] . \tag{12}$$

The integral in (12) represents retardation effects associated with wave guide cutoffs. It is expected to be largest at those (p, n) for which $A(p, n) = \alpha_p c - n\omega_0$ is small, giving a primarily reactive effect. The presence of the integral does not add a lot to the cost of a dynamical calculation, since one can store each of the integrals as a matrix $M(p, n)$, and update that matrix at each time step δt by adding the integral from t to $t + \delta t$. This requires a few floating point operations for each (p, n). Fig. 1 shows the function $\partial \tilde{Z}(n, n\omega_0)/\partial\omega$ that appears in the first order Taylor term, multiplied by ω_0. Parameters are for a compact storage ring studied in [3]: $R = 25$ cm , $h = 1$ cm , $E_0 = 25$ MeV.

In Fig.2 we report a first attempt at evaluating formula (12) in the context of a time-domain integration of the Vlasov equation. Beside the r.f. bucket, the force is entirely from (12). The machine parameters are those of [3]. The time parameter is $\tau = \omega_s t$, where ω_s is the circular synchrotron frequency. To model the smooth switching on of the current assumed above, we increase the charge from zero, multiplying the final charge Q by the function $f(\tau) = (1 - (\tau - 1)^4)^4$. The final charge at $\tau = 1$ is close to the value for onset of a CSR-induced instability ($I = 0.98$pC/V in the notation of [3]). We assume that

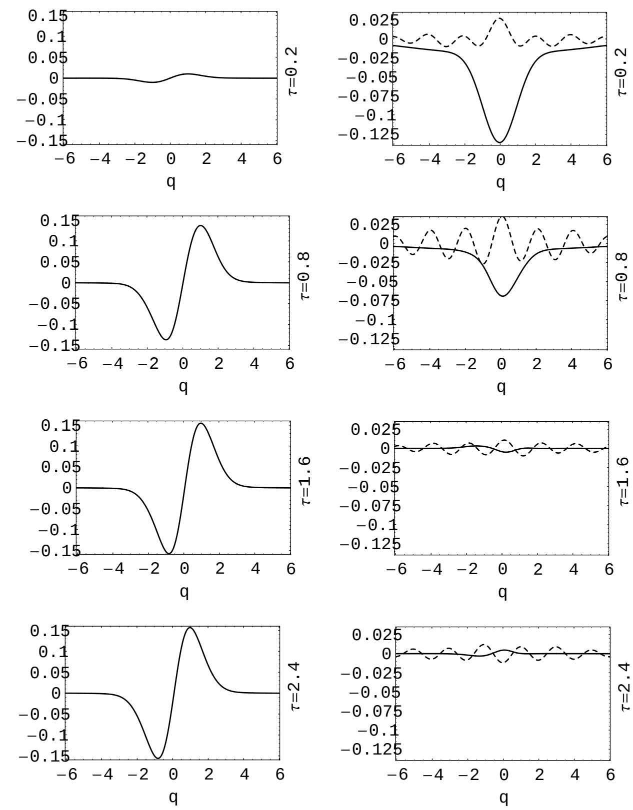

Figure 2: Dimensionless wake force $IF(q)$ (in notation of [3]) proportional to (12), at successive times. First term on left graph, second (solid) and third (dashed) on right. $q = (\theta - \omega_0 t)R/\sigma_z$, $\sigma_z =$ bunch length.

$\lambda_n'(t)$ is given by a simple divided difference, the time increment being that of the Vlasov integration. The curves on the left in Fig.2 are from the first term in (12) at successive times (up to about 4/10 of a synchrotron period). The solid and dashed curves on the right are from the second and third terms, respectively. The period of oscillations in the third term is exactly what one expects from peaking of the integral at n such that $A(1, n) = 0$. After initial transients the corrections to the first term are fairly small. It remains to be seen whether they remain small during full development of the instability, at somewhat later times.

REFERENCES

[1] R. Warnock and J. Ellison, in *Proc. 2nd ICFA Workshop on High Brightness Beams*, (World Sci., Singapore, 2000).

[2] M. Venturini and R. Warnock, Phys. Rev. Lett. **89**, 224802 (2002).

[3] M. Venturini, R. Warnock, and R. D. Ruth, to be submitted to Phys. Rev. **E**; R. Warnock and M. Venturini, in preparation.

[4] N. Towne, Phys. Rev. ST Accel. Beams **4**,114401 (2001).