NEW REGIONS OF STABILITY FOR PERIODICALLY FOCUSED PARTICLE BEAMS*

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Abstract

In this paper we perform a comprehensive analysis of the transport of periodically focused particle beams within the new regions of stability recently found [R. Pakter and F. B. Rizzato, Phys. Rev. Lett., **87**, 044801 (2001)] for vacuum phase advances well above the 90 degrees threshold. In particular, we investigate the stability as a function of the relevant parameters of the system, such as beam intensity and focusing field profile. Self-consistent numerical simulations are used to verify the findings.

INTRODUCTION

The physics of intense beams in periodically focusing systems is an active area of theoretical and experimental research where one looks for external field configurations capable of confining high-current, low emittance ion or electron beams [1, 2]. The area is crucial for the development of several advanced particle accelerator applications, as well as applications in basic science.

A key aspect of periodically focused beams is their equilibrium and stability properties. Up until recently, it was believed that only one matched solution – equilibrium solution where the beam transverse radius oscilates with the same periodicity of the focusing field – is present for a given set of beam and focusing parameters and that this solution becomes unstable as the focusing field intensity is raised above a certain threshold [1, 2]. The threshold corresponds to a vacuum-phase advance of 90 degrees. Recently, however, it was shown that new regions of stability that lead to much tighter beam confinement are present for vacuum-phase advances well above 90 degrees [3, 4]. In fact, the scenario as the focusing field increases is the appearance of successive regions of stability which are interrupted by gaps where either the matched solutions are unstable or simply do not exist. The dynamical mechanism responsible for the onset of the gaps is analyzed in Ref. [5].

In this paper, we perform a detailed investigation of beam envelope stability as a function of the relevant parameters of the system, such as beam intensity and focusing field profile.

THE MODEL

In the paraxial approximation the envelope equation that dictates the envelope evolution of a particle beam propagating with average axial velocity $\beta_b c\hat{\mathbf{e}}_z$ through a periodic solenoidal focusing magnetic field is given, in its dimensionless form, by

$$\frac{d^2 r_b}{ds^2} + \kappa_z(s) r_b - \frac{K}{r_b} - \frac{1}{r_b^3} = 0. \tag{1}$$

In Eq. (1), $s = z/S = \beta_b ct/S$ is the dimensionless coordinate along the beam axis, $r_b(s) = r_{b,dimensional}/(S\epsilon)^{1/2}$ is the normalized beam envelope radius and $K = 2q^2 N_b S/\epsilon\gamma_b^3\beta_b^2 mc^2$ is the normalized perveance of the beam where c is the speed of light in *vacuo*, S is the periodicity length of the magnetic focusing field, ϵ is the unnormalized emittance of the beam, N_b is the number of particles per unit axial length, and q, m and $\gamma_b = (1-\beta_b^2)^{-1/2}$ are, respectively, the charge, mass and relativistic factor of the beam particles. The focusing field is characterized by the normalized focusing strength parameter $\kappa_z(s+1) = \kappa_z(s)$ related to the magnetic field by $\kappa_z(s) = q^2 B_z^2(s) S^2/4\gamma_b^2\beta_b^2 m^2 c^4$.

In order to investigate the role of the focusing field profile on beam transport, we consider a focusing field parameter of the form

$$\kappa_z(s) = \sigma_0^2 \left[\frac{1+\cos\theta(s)}{N}\right], \tag{2}$$

with the phase function given by

$$\theta(s) = \pi \left\{ \frac{\tan^{-1}[\Delta(\bar{s}+\eta/2)] + \tan^{-1}[\Delta(\bar{s}-\eta/2)]}{\tan^{-1}[\Delta(1+\eta)/2] + \tan^{-1}[\Delta(1-\eta)/2]} \right\}, \tag{3}$$

where $\sigma_0 = [\int_0^1 \kappa_z(s) ds]^{1/2}$ is the vacuum phase advance in the smooth-beam approximation, which is proportional to the rms focusing field, $N = 1 + \int_0^1 \cos\theta(s) ds$ is used to normalize the function, $\bar{s} = \mathrm{mod}(s+1/2, 1) - 1/2$ is periodic in s and lies always in the range $-1/2 \geq \bar{s} \geq 1/2$, $\Delta > 0$ is the focusing field profile paramenter, and $0 < \eta \leq 1$ is the filling factor. The function $\kappa(s)$ in Eq. (2) is constructed such that for small Δ it resembles a smooth sinusoidal function of period 1 in s, while for increasing Δ it starts developing sharper edges, eventually turning into a discontinuous periodic step function of filling factor η for infinite Δ. In fact, in the limit $\Delta \ll 1$ the arguments of the inverse tangent functions in Eq. (3) are small, allowing the approximation $\tan^{-1}(x) = x$ which leads to $\theta(s) = 2\pi\bar{s}$ and to the sinusoidal focusing field profile $\kappa_z(s) = \sigma_0^2[1+\cos(2\pi s)]$, studied in Refs. [3, 4]. Note that in this limit, η plays no role in focusing field profile. On the other hand, when $\Delta \gg 1$ the inverse

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tangent functions present an abrupt change from $-\pi/2$ to $\pi/2$ as their arguments change sign, allowing the approximation $\tan^{-1}(x) = (\pi/2)\,\mathrm{sign}(x)$ which leads to a discontinuous phase function with $\theta(\bar{s} < -\eta/2) = -\pi$, $\theta(-\eta/2 < \bar{s} < \eta/2) = 0$ and $\theta(\bar{s} > \eta/2) = \pi$, and to a step-function focusing field lattice with filling factor η. Note that for all Δ, the denominator in Eq. (3) guarantees that the phase function completes a full cycle from $\theta = -\pi$ to $\theta = \pi$ as $\bar{s}$ goes from $-1/2$ to $1/2$, and consequently $\kappa_z(s)$ is always continuous at the lattice boundaries.

BEAM STABILITY ANALYSIS

In this section, we analyse the stability of beams propagating through the focusing field given in Eq. (2), as the relavant parameters of the system are varied, paying special attention to the new regions of stability. To perform the analysis we use a Newton-Raphson method to search for and verify the stability of envelope matched solutions obtained from Eq. (1). The stability is determined with the aid of the stability index α defined as $\alpha = \cos(k_{fix})$, where k_{fix} is the wavenumber of small linear oscillations around the periodic trajectory, obtained with the Newton-Raphson method. For stable orbits where k_{fix} is a real number, $|\alpha| < 1$; if α crosses the lower boundary $\alpha = -1$ it undergoes a period doubling bifurcation loosing stability, and if the orbit crosses the upper boundary $\alpha = +1$ the orbit undergoes an inverse tangent bifurcation with a previous unstable fixed point.

Generally, the bifurcation scenario for the matched solutions as one increases the vacuum phase advance is as follows; detailed description is found in Refs. [3, 4]. Stable matched solutions are born in the phase space with $\alpha = +1$. For the original matched solution, this occurs exactly at $\sigma_0 = 0^{\circ}$, whereas for the new regions of stability it occurs at different $\sigma_0 > 180^{\circ}$. As the vacuum phase advance is increased, the respective α moves towards $\alpha = -1$. When $\alpha = -1$ is reached, the matched solution suffers a period doubling bifurcation and becomes unstable. We define as a region of stability, the range of σ_0 that goes from the onset of the corresponding stable matched solution with $\alpha = +1$ until it bifurcates with $\alpha = -1$. As a matter of fact, before eventually dissapearing permanently from the phase space, the matched solutions cross back the $\alpha = -1$ line and recover their stability as σ_0 is further increased. However, as shown in Ref. [4] the matched solution is not useful for beam confinement after its re-stabilization because beam emitance growth was observed in self-consistent numerical simulations for these parameter regions.

To determine the role of a particular parameter on the beam transport stability we construct parametric space plots of σ_0 as a funtion of the parameter displaying the locations of the different regions of stability. The plots are obtained by using the Newton-Raphson method to numerically determine the curves in the parameter space for which a bifurcation with $\alpha = \pm 1$ occurs.

Dependence on the Beam Intensity

In Fig. 1 we analyze the dependence of the beam stabiliy on the beam intensity by ploting the parameter space plot of $K \times \sigma_0$, for a sinusoidal field with $\Delta \ll 1$. The white regions in the figure correspond to parameter values for which at least one stable matched solution with $|\alpha| < 1$ exists. The gray regions correspond to the existence of a single matched solution with $\alpha < -1$ which is unstable because a period doubling bifurcation has taken place. The black regions correspond to the gaps where no matched solution is found.

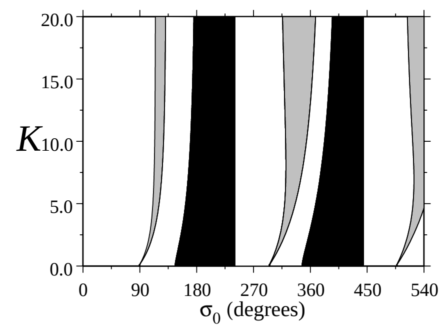

Figure 1: $K \times \sigma_0$ parameter space plot: stable regions are the white regions

Figure 1 shows that the perveance plays a tricky role on the characteristics of the new regions of stability. In particular, the size of the instability gap caused by the period doubling bifurcation (gray regions) tends to increase with increasing K. On the other hand, the size of the gap where matched solutions are absent (black regions) tends to decrease as K and/or n increases. Therefore, there is a particular values of K (around $K = 7.5$ for the specific sinusoidal focusing field used in Fig. 1) for which the new regions of stability are wider in the parameter space. Another interesting feature is that the onset of the new stable matched solutions, which are the bifurcations leading to the lines that limit the black regions to the right, is essentially independent of the perveance K. This can be understood based on the fact that the new matched solutions enter the phase-space as solutions that oscillate from $r_b = 0$ to $r_b \to \infty$ [4]. Because in this case the particles of the beam spend most of the time far away from each other, space charge effects introduced by K are unimportant.

Dependence on the Focusing Field Profile

Recalling from the model, Δ determines the overall shape of the focusing field: as Δ is increased from small values $\Delta \ll 1$, the focusing field profile continuously goes from a smooth sinusoidal function to a sharp-edged step-function as $\Delta \to \infty$. In Fig. 2, it is shown the parameter space plot of $\sigma_0 \times \Delta$, for a *small* filling factor $\eta = 0.2$, and a beam intensity corresponding to $K = 5.0$. The black regions correspond to stable regions; the region below $\sigma_0 \approx 100^o$ is the original region of stability (ORS) and the other one is the second region of stability (SRS). Here, we consider as stable regions only those preceeding the respective period doublings. Higher order new regions of stability were also investigated and the results are qualitetively the same as for the SRS.

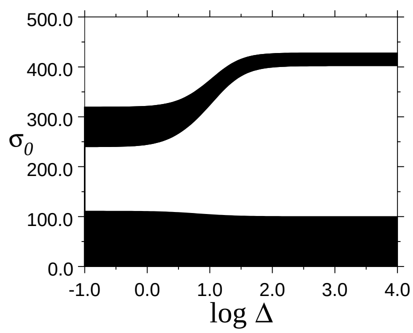

Figure 2: $\sigma_0 \times \Delta$ parameter space plot: stable regions are the black regions.

It is seen in Fig. 2 that the focusing field profile plays an important role in the beam stability and as Δ increases two effects are clearly seen regarding the SRS. (i) First, there is an increase in the vacuum phase advance necessary to reach the SRS. Since σ_0 is proportional to the rms focusing field, this reveals that the peak magnetic field has to be raised considerably as the profile becomes more localized with *small* η not only because its average has to increase, but also because the spatial region where the field is effectively applied is smaller. In the case depicted in Fig. 2(b), for instance, taking into account that σ_0 for the SRS increases roughly 50% as Δ goes from 10^{-1} to 10^4, the increase in the peak magnetic field has to be about 3.75 times. However, if one now looks at the minimum value attained by the matched beam envelope as it oscilates in the focusing lattice one notes that it is noticeably reduced as the peak magnetic field for the SRS increases with Δ. (ii) Second, the SRS becomes much narrower as Δ is increased. The range in vacuum phase advance for which the SRS exists goes from 80^o to 25^o as Δ is increased. Not only this reveals that a more accurate field intensity tuning is necessary as the focusing channel becomes more localized for $\eta = 0.2$, but it also suggests that more nonlinear resonances may appear in the phase space because the variation of α with σ_0, and hence the range of orbital frequencies in the phase space, is larger. The resonances may affect the beam transport nonlinear stability.

Other values of the filling factor η were also investigated and the overall conclusion is that $\eta = 0.5$ may be seen as a midpoint in the sense that at this value the new regions of stability are not greatly affected by the variations in the focusing field profile parameter. This is probably connected to the fact that exactly at $\eta = 0.5$ the sinusoidal ($\Delta \rightarrow 0$) and the step-function ($\Delta \rightarrow \infty$) limits of the focusing profile present the same norm $N = 1 + \int_0^1 \cos\theta(s)ds = 1.0$, such that the peak magnetic field is the same in both cases. For $\eta > 0.5$ it was found that the new regions tend to increase in size as Δ is increased from 0, getting closer to the ORS. In fact, one may eventually find parameter sets for which two stable matched solutions coexist in the phase space. On the other hand, as shown in detail for $\eta = 0.2$, when $\eta < 0.5$ the new regions become narrower and occur at higher vacuum phase advances as Δ is increased. In particular, for the thin lens regime where $\eta \rightarrow 0$, $\Delta \rightarrow \infty$, and $\kappa_z(s)$ tends to a series of Dirac-delta functions, the onset of the new regions of stability only occur at $\sigma_0 \rightarrow \infty$, which in practice means that these regions are absent. However, this limit is not realistic due to the restrictions imposed by Maxwell's equations on the focusing field profile.

CONCLUSIONS

The new regions of stability found for vacuum phase advance well above 90 degrees are sensitive to variations in the relevant parameters of the system, specially to variations in the shape of the focusing field profile. However, they are always present and are robust against parameter changes.

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