CALCULATION OF THE COHERENT SYNCHROTRON RADIATION IMPEDANCE FROM A WIGGLER*

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Abstract

Most studies of coherent synchrotron radiation (CSR) have considered only the radiation from independent dipole magnets. However, in the damping rings of future linear colliders and many high luminosity factories, a large fraction of the radiation power will be emitted in damping wigglers. In this paper, the longitudinal wakefield and impedance due to CSR in a wiggler are derived in the limit of a large wiggler parameter K. After an appropriate scaling, the results can be expressed in terms of universal functions, which are independent of K. Analytical asymptotic results are obtained for the wakefield in the limit of large and small distances, and for the impedance in the limit of small and high frequencies.

INTRODUCTION

Many modern advanced accelerator projects call for short bunches with low emittance and high peak current where coherent synchrotron radiation (CSR) effects may play an important role. CSR is emitted at wavelengths longer than or comparable to the bunch length whenever the beam is deflected [1]. The stringent beam requirements needed for short wavelength Self-Amplified Spontaneous Emission (SASE) free-electron lasers have led to intensive theoretical and experimental studies [2] over the past few years where the focus has been on the magnetic bunch compressors required to obtain the high peak currents. In addition to these single-pass cases, it is also possible that CSR might cause a microwavelike beam instability in storage rings. A theory of such an instability in a storage ring has been recently proposed in Ref. [3] with experimental evidence published in [4].

The previous study of the CSR induced instability assumed that the impedance is generated by the synchrotron radiation of the beam in the storage ring bending magnets [3]. In some cases (e.g. the Next Linear Collider (NLC) damping ring [5]), a ring will include magnetic wigglers which introduce an additional contribution to the radiation impedance. The analysis of the microwave instability in such a ring requires knowledge of the impedance of the synchrotron radiation in the wiggler. Although there have been earlier studies of the coherent radiation from a wiggler or undulator [6, 7], the results of these papers cannot be used directly for the stability analysis.

In this paper, we derive the CSR wake and impedance for a wiggler. We focus our attention on the limit of a large wiggler parameter K because this is the most interesting case for practical applications. It also turns out that, in this limit, the results can be expressed in terms of universal functions of a single variable after an appropriate normalization.

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ENERGY LOSS AND LONGITUDINAL WAKE IN WIGGLER

The longitudinal wake is directly related to the rate of energy loss $d\mathcal{E}/dt$ of an electron in the beam propagating in a wiggler. For a planar wiggler, a general expression for $d\mathcal{E}/dt$ as a function of the position s of the electron in the bunch and the coordinate z in the wiggler was derived in Ref. [7]. We reproduce here the results of that work using the authors' notation:

$$\frac{d\,\mathcal{E}}{c\,dt} = e^2\,k_w \int_{-\infty}^{s} d\,s'\,D(\hat{s}-\hat{s}',K,\hat{z})\frac{d\,\lambda(s')}{ds'}, \qquad (1)$$

where $\lambda(s)$ is the bunch linear density,

$$D(\hat{s},K,\hat{z}) = \frac{1}{\hat{s}} - 2\times$$
$$\frac{\Delta - K^2\,B(\Delta,\hat{z})\,[\sin\Delta\cos\hat{z} + (1-\cos\Delta)\,\sin\hat{z}]}{\Delta^2 + K^2\,B^2(\Delta,\hat{z})}, \qquad (2)$$

$$\begin{aligned} B(\Delta,\hat{z}) = \; & (1 \;-\cos\Delta - \Delta\sin\Delta)\cos\hat{z} \\ + \; & (\Delta\cos\Delta - \sin\Delta)\sin\hat{z}, \end{aligned} \qquad (3)$$

and Δ is the solution of the transcendental equation

$$\begin{aligned} \hat{s} \;\; = \;\; & \frac{\Delta}{2}\left(1+\frac{K^2}{2}\right) + \frac{K^2}{4\,\Delta}\{[2(1-\cos\Delta) - \Delta\sin\Delta] \\ \times \;\; & (\cos\Delta\cos 2\hat{z} + \sin\Delta\sin 2\hat{z}) - 2(1-\cos\Delta)\}. \end{aligned} \qquad (4)$$

In the above equations, we use the following dimensionless variables: $\hat{s} = \gamma^2\,k_w\,s$ and $\hat{z} = k_w\,z$. The parameter Δ is equal to $k_w\,(z - z_r)$, where z and z_r are the projected coordinates on the wiggler axis of the current position of the test particle and the retarded position of the source particle, respectively. The internal coordinate s is defined so that the bunch head corresponds to a larger value of s than the tail. The wiggler parameter K is approximately $K \approx 93.4\,B_w\,\lambda_w$, with B_w the peak magnetic field of the wiggler in units of Tesla and λ_w the period in meters. In addition, γ is the Lorentz factor, e is the electron charge, c is the speed of light in vacuum, and $k_w = 2\pi/\lambda_w$ is the wiggler wave number. Note that the function D is a periodic function of $\hat{z}$ with a period equal to π. Also note that, despite assuming $K \gg 1$, we still assume a small-angle orbit approximation, i.e., $K/\gamma \ll 1$.

We introduce the longitudinal wake $W(s)$ of the bunch as the rate of the energy change averaged over the z coordinate:

$$W(s) = -\frac{1}{e^2}\frac{d\,\bar{\mathcal{E}}}{c\,dt} = -k_w\int_{-\infty}^{s} ds' G(s-s')\frac{d\,\lambda(s')}{ds'}, \qquad (5)$$

where $$G(s) = \frac{1}{\pi}\int_0^{\pi} d\,\hat{z}\, D(\hat{s}, K, \hat{z}), \tag{6}$$

and we dropped K from the list of arguments of the function G. The positive values of W correspond to the energy loss and the negative values imply the energy gain. The usual longitudinal wake $w(s)$ corresponding to the interaction of two particles is then defined as

$$w(s) = -k_w\, \frac{d\,G(s)}{d\,s}\ , \tag{7}$$

so that $$W(s) = \int_{-\infty}^{s} ds'\, w(s - s')\lambda(s'). \tag{8}$$

Note that the wake Eq. (7) is localized in front of the particle and vanishes behind it; $w = 0$ for $s < 0$.

In the limit of large K, we can neglect unity in the first bracket of Eq. (4), assuming that $K^2/2 \gg 1$. Such an approximation is valid, if we are not interested in the very short distances of order of $(K\, k_w\, \gamma^2)^{-1}$ (0.5Å for the NLC damping ring wiggler [5]). We also introduce a new variable $\zeta \equiv \hat{s}/K^2$ which eliminates the parameter K from Eq. (4):

$$\zeta(\Delta, \hat{z}) = \frac{\Delta}{4} + \frac{1}{4\Delta}\{[2(1-\cos\Delta) - \Delta\sin\Delta] \times(\cos\Delta\cos 2\hat{z} + \sin\Delta\sin 2\hat{z}) - 2(1-\cos\Delta)\}. \tag{9}$$

In this limit, the expression for D, Eq. (2), can also be simplified:

$$D(\zeta, \hat{z}) = 2\,\frac{\sin\Delta\cos\hat{z} + (1-\cos\Delta)\sin\hat{z}}{B(\Delta, \hat{z})}, \tag{10}$$

as long as Δ is not too small, $\Delta \gg 1/K$. Again, the parameter K is eliminated from this equation.

WAKEFIELD

Using Eq. (6) and (10) we find

$$G(\zeta) = \frac{2}{\pi}\int_0^{\pi} d\,\hat{z}\,\frac{\sin\Delta\cos\hat{z} + (1-\cos\Delta)\sin\hat{z}}{B(\Delta, \hat{z})}, \tag{11}$$

where $\Delta = \Delta(\zeta, \hat{z})$ is implicitly determined by Eq. (9). The integrand in this equation has singularities at points $\hat{z} = \hat{z}_s$ where $B(\Delta(\zeta, \hat{z}_s), \hat{z}_s) = 0$. It could be checked that in the vicinity of a singular point $B(\Delta(\zeta, \hat{z})) \propto (\hat{z} - \hat{z}_s)^{1/3}$, and the singularity is integrable.

We plot the function $G(\zeta)$ calculated by numerical integration in Fig. 1. A characteristic feature of the function G is the presence of cusp points, at which the function reaches local maxima and minima. Approximately, they are

$$G(\zeta) = \begin{cases} -\frac{4\,(2\,n-1)\,\pi}{4+[(2\,n-1)\,\pi]^2}; & \text{at } \zeta \approx \frac{(2n-1)\,\pi}{4} - \frac{1}{(2n-1)\,\pi} \\ 0; & \text{at } \zeta = \frac{n\,\pi}{2} \end{cases} \tag{12}$$

with $n = 1, 2, \cdots$. These are the "$\times$" points in Fig. 1, showing very good agreement with the numerical result. The longitudinal wake given in Eq. (7) will reach infinity when approaching the maxima and minima from one side, and negative infinity on the other side.

In the limit $\zeta \ll 1$, it follows from Eq. (9) that $\Delta \ll 1$ as well. Equation (9) can then be solved using a Taylor expansion of the right-hand side: $\Delta = \left(24\,\zeta/\cos^2\hat{z}\right)^{1/3}$. Expanding the integrand in Eq. (11), keeping only the first nonvanishing term in Δ yields

$$G(\zeta) = -\frac{1}{\pi}\frac{2}{(3\,\zeta)^{1/3}}\int_0^{\pi} d\hat{z}\cos^{2/3}\hat{z} = -\frac{4\,3^{2/3}\,\Gamma\left(\frac{11}{6}\right)}{5\sqrt{\pi}\,\Gamma\left(\frac{4}{3}\right)}\,\zeta^{-1/3} \approx -0.99\,\zeta^{-1/3}. \tag{13}$$

The above result can also be obtained if one considers a wiggler as a sequence of bending magnets with the bending radius $R = \gamma/k_w K|\cos\hat{z}|$, since in this limit the formation length of the radiation is much shorter than the wiggler period, and one can use a local approximation of the bending magnet for the wake. In the limit $\zeta \gg 1$, the parameter Δ

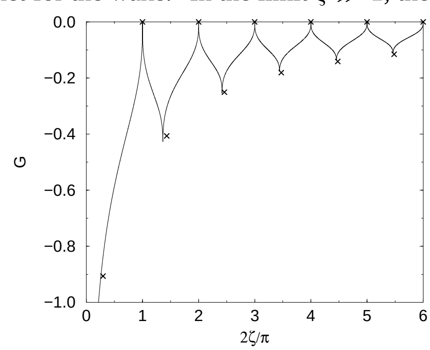

Figure 1: The solid curve represents the $G(\zeta)$ defined in Eq. (11) as a function of the normalized coordinate $2\zeta/\pi$. The $(\times)$ signs are the approximation given in Eq. (12).

is also large, and Eq. (9) can be further simplified:

$$\zeta = \frac{\Delta}{4} - \frac{\sin\Delta\,\cos(\Delta - 2\hat{z})}{4}. \tag{14}$$

In Eq. (3), we keep only the largest term $B(\Delta, \hat{z}) = -\Delta\,\sin(\Delta - \hat{z})$. For D, one now finds, $D(\zeta, \hat{z}) \equiv F(\zeta, \hat{z})/\zeta$, with

$$F(\zeta, \hat{z}) \equiv \frac{\sin\hat{z}}{2\,\sin(\hat{z} - \Delta(\zeta, \hat{z}))} - \frac{1}{2}, \tag{15}$$

where the function $\Delta(\zeta, \hat{z})$ is implicitly determined by Eq. (14). Averaging over one wiggler period, we find $G(\zeta) \equiv \bar{F}(\zeta)/\zeta$, with

$$\bar{F}(\zeta) \equiv \frac{1}{\pi}\int_0^{\pi} d\hat{z} F(\zeta, \hat{z}) = \frac{1}{2\,\pi}\left(-\pi + \int_0^{\pi} d\hat{z}\frac{\sin\hat{z}}{\sin(\hat{z} - \Delta)}\right). \tag{16}$$

It is easy to check that the function $\bar{F}$ is periodic, $\bar{F}(\zeta + \pi/2) = \bar{F}(\zeta)$, and $\bar{F}(0) = 0$, $\bar{F}(\pi/4) = -1$ in agreement with Eq. (12). The average value $\bar{F}(\zeta)$ is equal to $-1/2$. Since $\bar{F}$ is periodic in ζ with a period of $\pi/2$, using Eq. (16), we can obtain a Fourier series representation for $\bar{F}(\zeta)$. The corresponding long-range wake is then

$$G(\zeta) = -\frac{1}{2\zeta} + \frac{1}{2\zeta}\sum_{n=0}^{\infty}\left[J_n\!\left(\frac{2n+1}{2}\right) - J_{n+1}\!\left(\frac{2n+1}{2}\right)\right]^2 \cos(4(2n+1)\zeta). \tag{17}$$

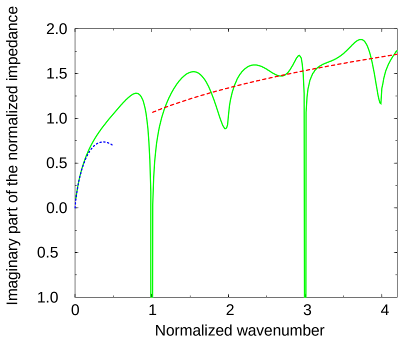

Figure 2: The imaginary part of the normalized impedance $Z(k)/k_w$ as a function of the normalized wave number k/k_0. Solid line: numerical solution from Eq. (18); dotted line: analytical low-frequency asymptotic behavior from Eq. (19); and dashed line: analytical high-frequency asymptotic behavior from Eq. (20).

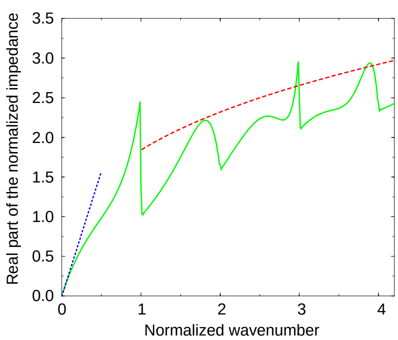

Figure 3: The real part of the normalized impedance $Z(k)/k_w$ as a function of the normalized wave number k/k_0. Solid line: numerical solution from Eq. (18); dotted line: analytical low-frequency asymptotic behavior from Eq. (19); and dashed line: analytical high-frequency asymptotic behavior from Eq. (20).

IMPEDANCE

The impedance $Z(k)$ is defined as the Fourier transform of the wake,

$$Z(k)=\int_0^\infty ds w(s)e^{-iks}=\frac{-ikK^2}{\gamma^2}\int_0^\infty d\zeta G(\zeta)e^{-4i(k/k_0)\zeta}, \quad (18)$$

where $k_0 \equiv 4\gamma^2 k_w/K^2$ is the wiggler fundamental radiation wave number.

We evaluated the integral in Eq. (18) using numerically calculated values of the function $G(\zeta)$ in the interval $[\zeta_{\min}, \zeta_{\max}]$, where $\zeta_{\min} \approx 10^{-3}$ and $\zeta_{\max} \approx 50$. The contribution to the integral outside of this interval was calculated using asymptotic representations Eqs. (13) and (17). The resulting imaginary and real parts of the impedance are shown in Figs. 2 and 3, respectively.

Simple analytical formulas for the impedance can be obtained in the limit of low and high frequencies. The low-frequency impedance corresponds to the first term in Eq. (17) for function G which does not oscillate with ζ: $G(\zeta) = -1/(2\zeta)$. Using the definition in Eq. (18), we then obtain the low-frequency asymptotic behavior of the impedance as

$$\begin{aligned} Z(k) &= -i\,2\,k_w\frac{k}{k_0}\left[\gamma_{\mathrm{E}} + \log\left(\frac{4k}{k_0}\right) + i\frac{\pi}{2}\right] \\ &\approx \pi\,k_w\frac{k}{k_0}\left[1 - \frac{2\,i}{\pi}\log\left(\frac{k}{k_0}\right)\right], \end{aligned} \quad (19)$$

where, $\gamma_{\mathrm{E}} \approx 0.5772$ is the Euler gamma constant. Since we have an analytical expression for the short-range $G(\zeta)$ in Eq. (13), we get the asymptotic high-frequency impedance as

$$\begin{aligned} Z(k) &= -i\frac{6\,\Gamma\left[\frac{11}{6}\right]}{5\sqrt{\pi}\,\Gamma\left[\frac{4}{3}\right]}A\left(\frac{Kk_w}{\gamma}\right)^{2/3}k^{1/3} \\ &\approx -0.71\,i\,A\left(\frac{Kk_w}{\gamma}\right)^{2/3}k^{1/3}, \end{aligned} \quad (20)$$

with $A = 3^{-1/3}\Gamma(2/3)(\sqrt{3}\,i - 1) \approx 1.63\,i - 0.94$ [3]. The asymptotic low- and high-frequency impedance are plotted in Figs. 2 and 3 for comparison with the numerical solution.

DISCUSSION AND CONCLUSION

In this paper, we derived the wakefield and the impedance for wigglers with $K^2/2 \gg 1$ due to the synchrotron radiation. Analytical asymptotic results are obtained for the wakes in the limit of small and large distances and for the impedance in the limit of small and high frequencies. The results obtained in this paper are used for the beam instability study due to the synchrotron radiation in wigglers [8].

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