A NUMERICAL MODEL OF ELECTRON BEAM SHOT NOISE

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Abstract

A numerical method of simulating electron beam shot noise in free electron lasers is presented. The method uses a quasi-uniform phase space distribution of appropriately charge weighted macroparticles. The statistical properties of the macroparticles are derived directly from the temporal Poisson statistical properties of the real electron distribution. Unlike previous methods, our method does not rely upon any averaging over a resonant radiation period timescale and so more correctly describes the underlying physics. The method also allows shot noise to be modelled self-consistently in un-averaged FEL models which are able to describe sub-wavelength phenomena such as Coherent Spontaneous Emission.

INTRODUCTION

Any computer code that attempts to model Self Amplified Spontaneous Emission in a FEL, or any other electron beam source, should have a valid numerical model of the electron beam shot-noise. In order to model shot-noise, a macroparticle distribution should simulate the statistical properties of the real electron distribution. Many, if not most simulation codes use the algorithm of [1] as the basis of their macroparticle loading algorithm to describe shot-noise. However, this algorithm, although effective, has been derived directly from the statistical properties of an averaged quantity, b, the bunching parameter [2]. This averaging occurs over a resonant radiation period. There has not been, to the authors' knowledge, a contiguous derivation from the statistical properties of the individual electrons in an electron beam to the algorithm of [1]. A noise model derived directly from the properties of the individual electrons would also allow the introduction of shot-noise into FEL models that have not been averaged over a radiation period in a consistent way. Although the method of [1] has been used in unaveraged models such as [3] it cannot be considered consistent to use methods developed from averaged equations in an unaveraged model. The work presented here derives such a macroparticle model directly from a Poisson statistical electron distribution [4].

THE MODEL

The arrival of electrons at the beginning of an interaction region, $z = 0$, is assumed to be a Poisson process. We first discretise time into small intervals of uniform duration Δt so that $t_n = n\Delta t$ where $n = 0, \pm 1, \pm 2, ...$ In the notation hereafter subscript n always refers to these discrete times. It will be seen that the time interval Δt is the mean interval between macroparticles introduced to model the real electron distribution. Furthermore, Δt is small with respect to any radiation period to be subsequently modelled i.e. $\Delta t \ll 2\pi/\omega_{\max}$.

Consider the arrival of electrons over one such time interval $t_n \leq t < t_{n+1}$. The mean rate of the Poisson process is the rate of electron arrival $\nu_n = I(t_n)/e$ which is assumed constant over the interval Δt. The electron arrival times obey Poisson statistics and the number of electrons, N_n, arriving within the interval Δt is a statistical variable determined by the Poisson distribution:

$$P(N_n) = \frac{\bar{N}_n^{N_n} e^{-\bar{N}_n}}{N_n!} \qquad (1)$$

where $\bar{N}_n = \nu_n \Delta t$, is the expectation for the number of electrons in the interval Δt. It can been shown [5] that the ordered arrival times of the electrons have identical statistical properties to those with unordered arrival times each of which have been distributed within the interval Δt with an identical uniform probability density $p_n = \nu_n/\bar{N}_n = 1/\Delta t$. The statistics of variables distributed with uniform probability density over a finite interval are well known [7], from which we obtain the mean and variance of each of the unordered electron arrival times t_j, $j = 1..N_n$ to be

$$\begin{aligned} \mu_n &= t_n + \Delta t/2 \quad \text{and} \\ \sigma_n^2 &= \Delta t^2/12 \end{aligned} \qquad (2)$$

respectively.

For a total of N_n electrons the mean arrival time is given by

$$\bar{t}_n = \frac{1}{N_n}\sum_{j=1}^{N_n} t_j$$

with expectation and variance of $\bar{t}_n$ easily shown to be

$$\begin{aligned} E(\bar{t}_n) &= \bar{\mu}_n = \mu_n; \quad \text{and} \\ V(\bar{t}_n) &= \bar{\sigma}_n^2 = \frac{\sigma_n^2}{N_n} = \frac{\Delta t^2}{12 N_n} \end{aligned} \qquad (3)$$

respectively.

The distribution of the N_n electrons within the interval $t_n \leq \Delta t \leq t_{n+1}$ may now be modelled by replacing the distribution with a single macroparticle whose statistical properties of charge and temporal distribution equal those of the N_n electrons. This is the physical basis of the model presented here.

The statistical properties of the macroparticle charge are given simply by the Poisson distribution (1). When loading the macroparticles in a numerical simulation code each would be assigned a charge weight of $Q_n = N_n e$ where e is the charge of an electron and N_n, the macroparticle electron number, would be generated by a Poisson random deviate generator of mean $\bar{N}_n$.

The statistical properties of the macroparticle arrival time may be found by firstly placing the macroparticle at the mean electron distribution arrival time $\bar{\mu}_n$. This mean arrival time then has added to it an independent random variable τ with uniform probability distribution over the interval $[-\delta t/2, \delta t/2]$. The interval δt is chosen so that the variance in the macroparticle arrival time is equal to that of the real electron distribution (3). Similarly to relation (2) the variance for the macroparticle arrival time is $\delta t^2/12$. Equating this macroparticle variance to that of the real electron distribution (3) the following relation for δt is obtained

$$\delta t = \frac{\Delta t}{\sqrt{N_n}} \approx \frac{\Delta t}{\sqrt{\bar{N}_n}} \tag{4}$$

where the latter approximation may be used when $\bar{N}_n \gg 1$.

An electron pulse defined by a mean current $I(z,t)$ may be modelled to include the effects of shot-noise by assigning macroparticles, with charge and arrival time as described above, over many such consecutive time intervals defined by t_n. The macroparticle arrival times describing the complete electron pulse may then be written as

$$t_j = \bar{t}_j + \tau_j, \quad j = 1..N_m \tag{5}$$

where N_m is the total number of macroparticles and the $j=1$ macroparticle arrives in the nth time interval so that $\bar{t}_j = t_n + (j-1/2)\Delta t$. For consistency, the Poisson variate N_n, the electron number for the jth macroparticle, is now written N_j.

It is important to note that the electron distribution and its statistics have been modelled by the macroparticles without reference to any external lengths or timescales such as a resonant radiation frequency or its harmonics. In this sense this analysis is self-consistent.

BUNCHING STATISTICS IN 1-D

The following scaled form of the 1-D wave equation describing the FEL interaction may be derived:

$$\left(\frac{\partial}{\partial \bar{z}} + \frac{\partial}{\partial \bar{z}_1}\right) A(\bar{z}, \bar{z}_1) = \frac{1}{\bar{n}_{p\|}} \sum_{j=1}^{N} \exp\left(-i\frac{\bar{z}_1}{2\rho}\right) \delta(\bar{z}_1 - \bar{z}_{1j}). \tag{6}$$

Details of the scaling may be found in [6]. The right hand side of the wave equation is written in terms of the real electron distribution, where N is the total number of electrons in the electron pulse and at the beginning of the interaction region, $z = 0$, so that $\bar{z}_1 = -2\rho\omega t$ where ρ is the FEL scaling parameter [2]. Note that the total number of electrons in the electron pulse, N, is itself a Poisson variate of mean $\bar{N}$, the expectation value of the total electron number given by

$$\bar{N} = \int_{-\infty}^{\infty} \frac{I(t)}{e}\, dt. \tag{7}$$

The Dirac delta function transforms to real units as

$$\delta(\bar{z}_1 - \bar{z}_{1j}) = \frac{\delta(\omega t - \omega t_j)}{2\rho}. \tag{8}$$

and the electron distribution may be replaced by a macroparticle distribution as described in the previous section to give the right hand side of the wave equation (6) in terms of the macroparticle distribution

$$\frac{2\pi}{N_{m\lambda}} \sum_{j=1}^{N_m} \frac{N_j}{\bar{N}_{\mathrm{pk}}} e^{i\omega t} \delta(\omega t - \omega t_j) \tag{9}$$

where $N_{m\lambda} = 2\pi/\omega\Delta t$ is the number of macroparticles within one radiation period and $\bar{N}_{\mathrm{pk}} = I_{\mathrm{pk}}\Delta t/e$ is the expectation of the macroparticle electron number at the peak of the electron pulse current, I_{pk}.

When averaged over a radiation period the right hand side of the wave equation yields a quantity known as the 'bunching parameter' [2]. If the macroparticle model is valid then the statistics of the bunching parameter must be the same for the macroparticle distribution as for a real electron distribution, and we now test for this.

Averaging (9) by integrating over one radiation period centred at time t yields an expression for the localised macroparticle bunching parameter:

$$b(t) = \frac{1}{N_{m\lambda}} \sum_{j=1}^{N_{m\lambda}} \frac{N_j}{\bar{N}_{\mathrm{pk}}} e^{i\omega\tau_j} e^{i\omega\bar{t}_j}. \tag{10}$$

The delta function has extracted those macroparticles within the interval (we retain j as the index for simplicity) and we have used relation (5).

The expectation of the bunching parameter (10) is then given as

$$E(b) = \frac{1}{N_{m\lambda}} \sum_{j=1}^{N_{m\lambda}} E_{N_j}\left(\frac{N_j}{\bar{N}_{\mathrm{pk}}} E_{N_j|\tau_j}\left(e^{i\omega\tau_j}\right)\right) e^{i\omega\bar{t}_j} \tag{11}$$

where $E_{N_j}(...)$ signifies the expectation value with respect to the macroparticle electron number N_j and $E_{N_j|\tau_j}(...)$ signifies the expectation value with respect to the randomness of the macroparticle arrival time τ_j for a given value N_j. The latter expectation is obtained by averaging in

τ_j over the interval $[-\delta t/2, \delta t/2]$ from which using relation (4) the following result is obtained:

$$E_{\tau_j|N_j}\left(e^{i\omega\tau_j}\right) = \frac{\sqrt{N_j}N_{m\lambda}}{\pi}\sin\left(\frac{\pi}{\sqrt{N_j}N_{m\lambda}}\right). \quad (12)$$

Substituting for (12) into (11), using the result that for a Poisson distribution $E_{N_j}(N_j) = \bar{N}_j$ and assuming the usually easily satisfied condition $\sqrt{N_j}N_{m\lambda} \gg 1$ so that

$$\sin\left(\frac{\pi}{\sqrt{N_j}N_{m\lambda}}\right) \approx \frac{\pi}{\sqrt{N_j}N_{m\lambda}},$$

the expression for the expectation of the bunching is

$$E(b) = \frac{1}{N_{m\lambda}}\sum_{j=1}^{N_{m\lambda}}\frac{\bar{N}_j}{\bar{N}_{\mathrm{pk}}}e^{i\omega\bar{t}_j}. \quad (13)$$

A similar but more lengthy analysis for the expectation of $|b|^2$ may also be carried out to obtain

$$E\left(|b|^2\right) = \frac{1}{N_{m\lambda}^2\bar{N}_{\mathrm{pk}}^2}\sum_{j=1}^{N_{m\lambda}}\bar{N}_j + |E\left(b\right)|^2 \quad (14)$$

from which an expression for the variance of the bunching, $V(b) = E(|b|^2) - |E(b)|^2$, is obtained.

It can be seen from (13) for the expression for $E(b)$ that if the current is not uniform over the radiation period, i.e. $\bar{N}_j \neq$ constant $\forall\ j$, then $E(b)$ will be non zero. Such non-zero bunching is caused by a current gradient and is the source of Coherent Spontaneous Emission [6].

In the limit of a uniform current beam we may set $\bar{N}_j = \bar{N}_{\mathrm{pk}}\ \forall\ j$, $E(b) = 0$ and there is no CSE. In this case $\bar{N}_{m\lambda}\bar{N}_{\mathrm{pk}} = \bar{N}_\lambda$, the expectation of the total number of electrons in the radiation period, and we obtain from (14) the result that $E(|b|^2) = 1/\bar{N}_\lambda$, which for the single radiation period under consideration here, is in agreement with previous analysis of averaged Poisson statistical models [4].

HIGHER DIMENSIONS

In the above analysis, electron beam properties such as energy and transverse momentum spread were neglected. In order to describe these effects and include the effects of shot-noise, the above model must be extended to a multi-dimensional electron phase space.

Phase space is first discretised into elemental ‘volumes’ by discretising along each phase space ordinate in a method similar to that carried out for time in the previous section. The populating by electrons of each phase space ‘volume element’ at $z = 0$ is then assumed to be a Poisson process with each element being populated at a local Poisson rate given by

$$\nu(\boldsymbol{\alpha}, t) = \frac{I(t)}{e}f(\boldsymbol{\alpha}) \quad (15)$$

where $\boldsymbol{\alpha}$ is a generalised phase space coordinate (e.g. $\boldsymbol{\alpha} = (\mathbf{r}_\perp, \mathbf{p})$, the transverse coordinate and the momentum respectively) and $f(\boldsymbol{\alpha})$ is a normalised distribution function. Note that, in general, the distribution function itself may have a temporal dependence via $\boldsymbol{\alpha}$.

The same algorithm as was used for allocating the temporal noise of the previous section is now used for each phase space coordinate of the macroparticles. The macroparticles are placed at the ‘centre’ of each phase space volume element and have added to each of their phase space coordinates an independent random variable of uniform probability distribution. This random variable is equivalent to the τ_j of the previous section and, for a generalised ordinate α_k, will have a range

$$\left[-\frac{\Delta\alpha_k}{2\sqrt{\bar{N}_j}}, \frac{\Delta\alpha_k}{2\sqrt{\bar{N}_j}}\right] \quad (16)$$

where $\Delta\alpha_k$ is the discretisation interval and from (15)

$$\bar{N}_j = \nu(\boldsymbol{\alpha}, t)\Delta V_\alpha \Delta t = \frac{I(t)}{e}f(\boldsymbol{\alpha})\Delta V_\alpha \Delta t \quad (17)$$

where $\Delta V_\alpha = \prod_k(\Delta\alpha_k)$ is the elemental phase space volume. Each macroparticle will also have assigned to it a Poisson random variate electron number of mean $\bar{N}_j$.

CONCLUSIONS

The derivation of the shot-noise model presented here is perhaps more physically intuitive, and therefore appealing, than those used in current FEL simulation codes: The macroparticle properties of arrival time *and* charge are derived directly from the intrinsic Poisson statistical properties of the individual electron arrival times at the beginning of the interaction region. The model is therefore independent of any external factors such as a resonant radiation period. The algorithm has been tested successfully in a numerical code and these results will be presented in a forthcoming publication.

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