

# Coulomb Crystals in Storage Rings for Quantum Information Science

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NAPAC 2022



@BrookhavenLab

# Outline

- Quantum Information Science (QIS)
- Crystalline beams
- Storage Rings for QIS
- Circular Radio-Frequency Quadrupoles
- Challenges
- Entanglement
- Entanglement in Ion traps

# Quantum Information Science

# *Simulating Physics with Computers*

## Richard P. Feynman

Keynote talk, 1<sup>st</sup> conference on Physics and Computation, MIT, 1981 (International Journal of Theoretical Physics, 21: 467–488, 1982)

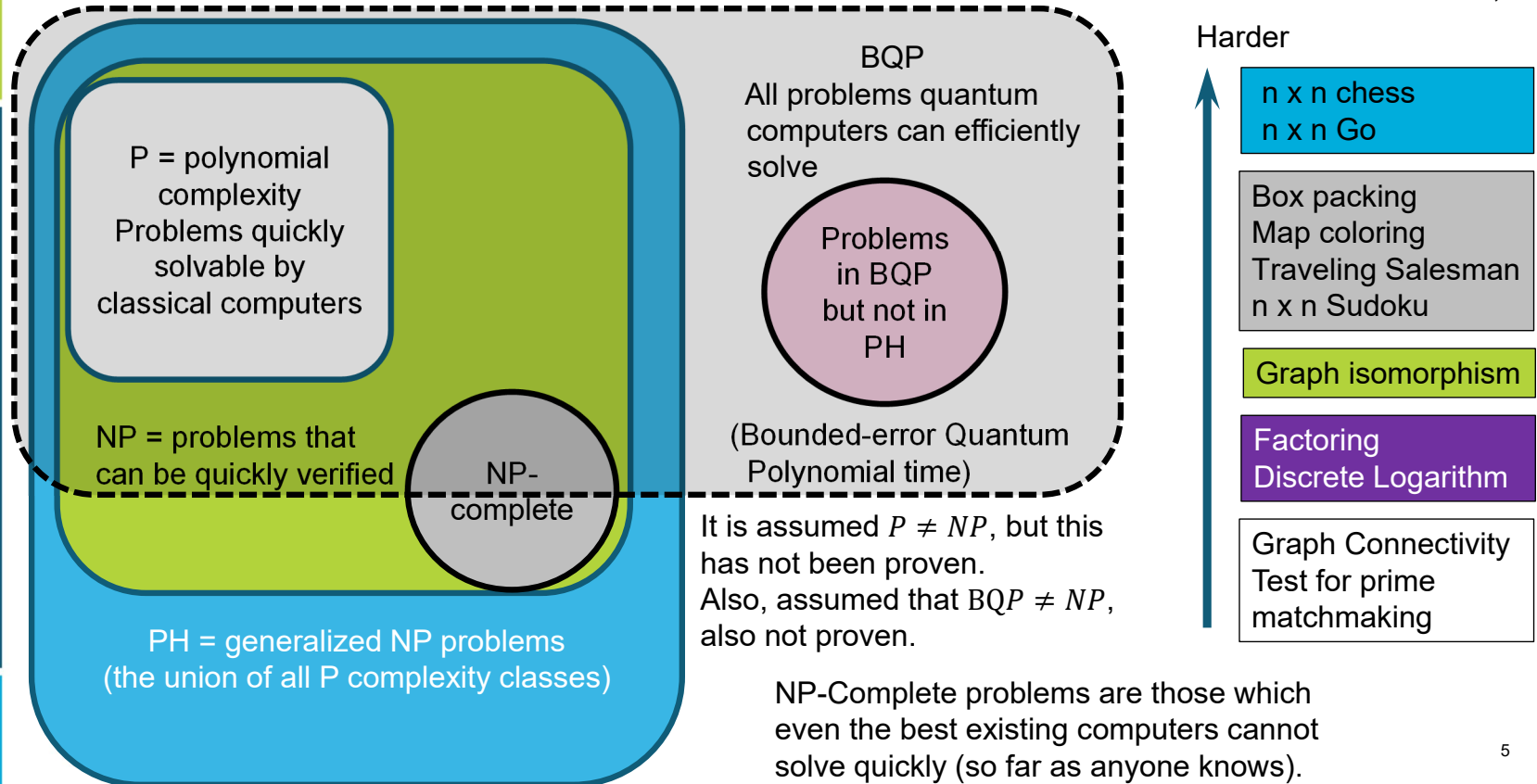
- Can a universal classical computer simulate physics exactly?
- Can a classical computer efficiently simulate quantum mechanics?
- "I'm not happy with all the analyses that go with just classical theory, because Nature isn't classical, dammit, and if you want to make a simulation of Nature, you'd better make it quantum mechanical, and by golly it's a wonderful problem!"

Richard Feynman 1981

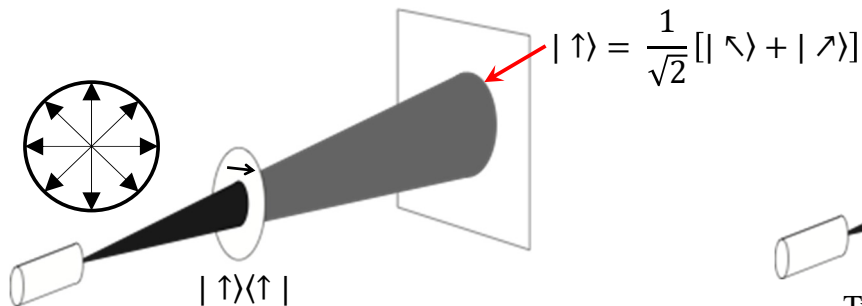


# The Limits of Quantum

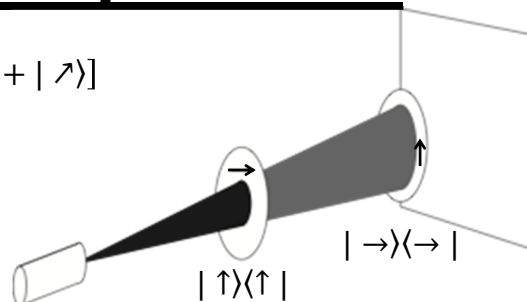
The Limits of Quantum  
By Scott Aaronson  
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# Dirac's Three Polarizers Experiment

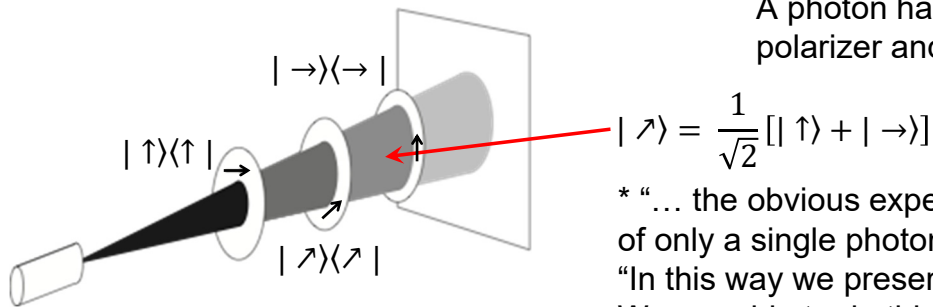


Single polaroid attenuates unpolarized light by 50 percent.



Two orthogonal polaroid's block all photons.

A photon has a probability  $\sin^2\alpha$  of passing through the polarizer and a probability  $\cos^2\alpha$  of being absorbed.



\* "... the obvious experiment is to use an incident beam consisting of only a single photon ..."

"In this way we preserve the individuality of the photon in all cases. We are able to do this, however, only because we abandon the determinacy of the classical theory."

Inserting a third polaroid allows photons to pass.

# Qubits

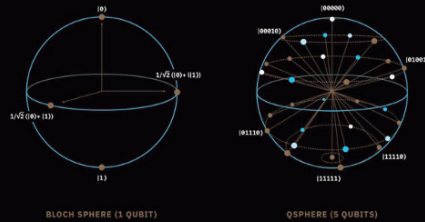
A quantum bit is quantum 2-level system.

Why is quantum different?

## 1. Superposition



Classical states



Quantum states

N qubits  
 $2^N$  paths

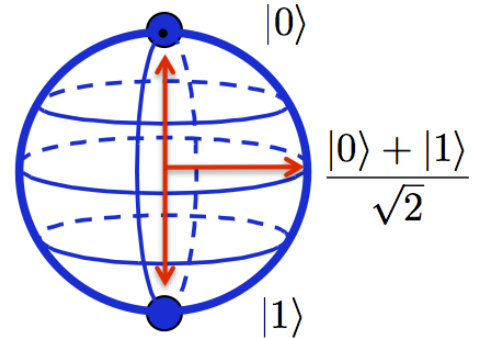


0



1

Classical Bit



Qubit

A measurement gives either a 0 or a 1

Between measurements it is in a superposition of both 0 and 1

$$|\varphi\rangle = a|0\rangle + b|1\rangle$$

$$\{a, b\} \in \mathbb{C}$$

$$P(\varphi \rightarrow 0) = |a|^2$$

<https://towardsdatascience.com/the-need-promise-and-reality-of-quantum-computing-4264ce15c6c0>

# How to make Qubits?

You need a quantum system or something that simulates a quantum system.

## 1. Quantum systems include

1. Ions in a trap
2. Quantum states in quantum materials (chiral magnetic effect)

## 2. Simulations of quantum systems include

1. Same as above (since an ion is used to simulate a quantum system)
2. Superconducting circuits such as Josephson junctions and Squids (transmon's)
3. Quantum annealing in a quantum material (limited applications, e.g. DWave)

## 3. Other approaches?

1. Trapped molecules in crystal lattices (e.g., flaws in diamonds)
2. Neutral trapped ions
3. Photonic systems

### References:

Christopher R. Monroe and David J. Wineland, *Quantum Computing with Ions*, © 2008 SCIENTIFIC AMERICAN, INC.

D. J. Wineland, et al., *Experimental primer on the trapped ion quantum computer*, Fortschr. Phys. 46, 363 (1998)

C. Monroe et al., *Programmable quantum simulations of spin systems with trapped ions*, Rev. of Mod. Phys., 93, April-June 2021

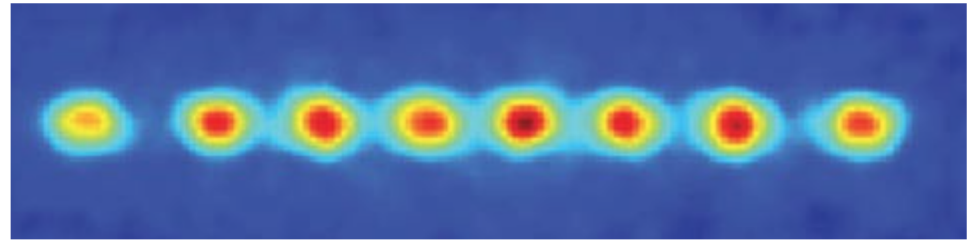
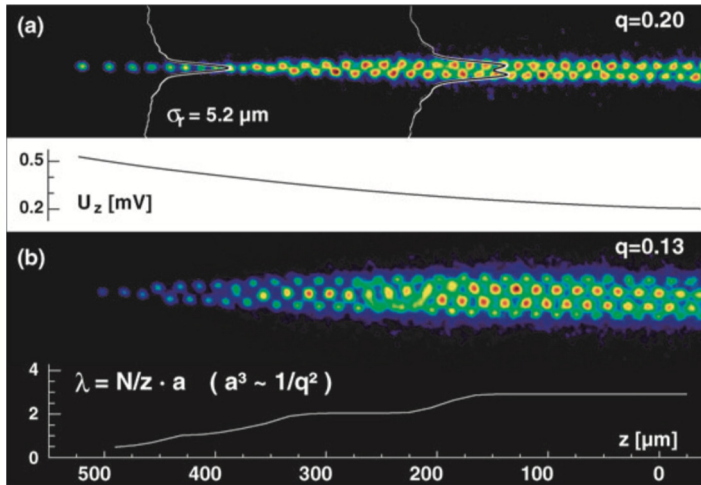
Kharzeev, Dmitri E. and Li, Qiang, *The Chiral Qubit: quantum computing with chiral anomaly*, <https://arxiv.org/abs/1903.07133>, 2019

A. Antony, et al., *Miniaturizing Transmon Qubits Using van der Waals Materials*, Nano Lett. 2021, 21, 23, 10122–10126

A. Chandra, A. Das, & B. Chakrabarti, eds. (2010). *Quantum Quenching, Annealing and Computation*. Lecture Note in Physics. Vol. 802

S. Pezzagna and J. Meijera, *Quantum computer based on color centers in diamond*, Applied Physics Reviews 8, 011308 (2021)

# Crystalline beams



Christopher R. Monroe and David J. Wineland

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- By using beam cooling techniques, ion beam temperatures can be reduced to the point that repelling Coulomb forces balance against external forces
- When this happens, a new form of matter is created, a chain of particles locked into a sequence

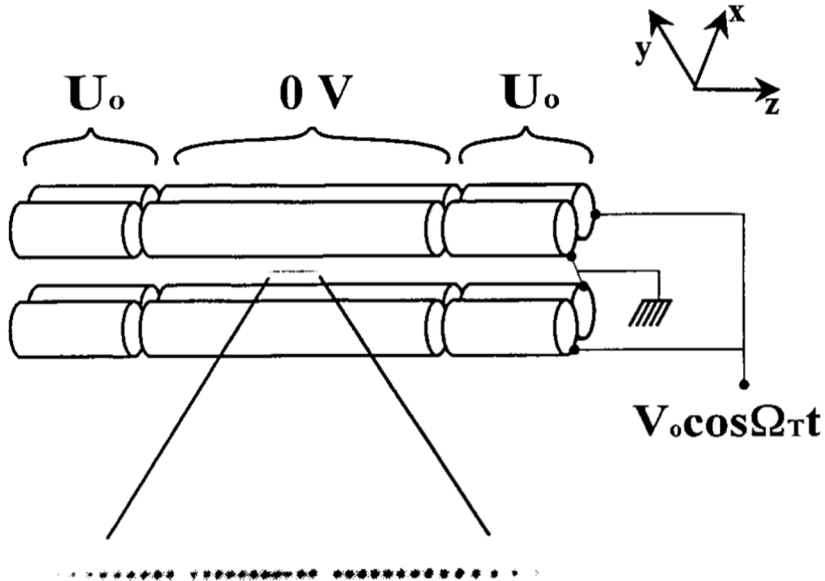
U Schramm, T Schätz, M Bussmann and D Habs  
 Plasma Phys. Control. Fusion **44** (2002) B375–B387

Sektion Physik, LMU München,  
 D-85748 Garching, Germany



# Storage Rings for QIS

# Ion Traps (or Paul Traps)



Earnshaw's theorem

one cannot construct a stable ion trap using electrostatic fields alone

Paul and Dehmelt got around Earnshaw's theorem by using oscillating electric fields, since the theorem strictly applies only to static fields.

*potential  $V_0 \cos \Omega_T t$  is applied between diagonally opposite rods, fixed in a quadrupole configuration*

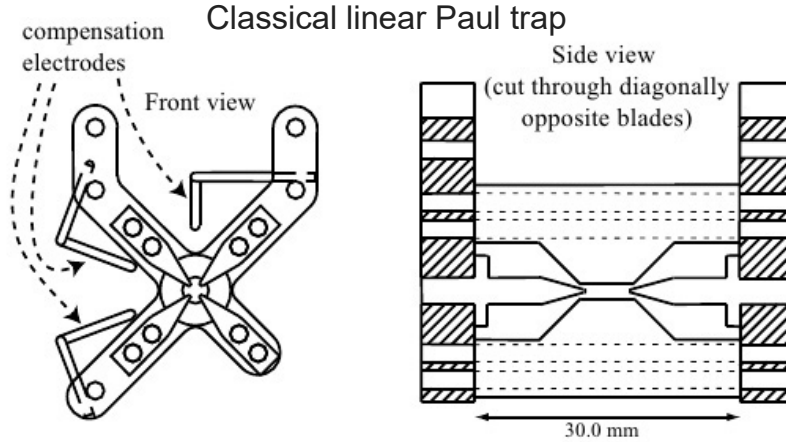
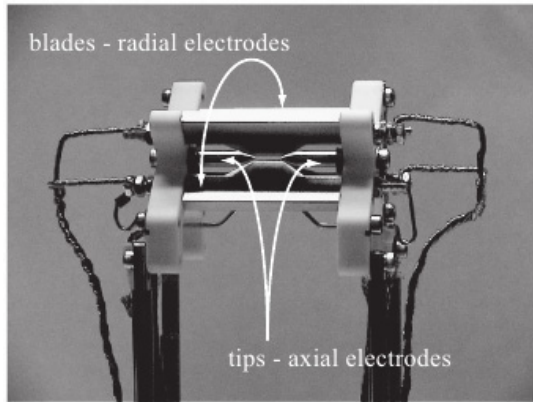
static potentials  $U_0$  are applied to the end segments

RF frequency must be high enough such that E-field oscillates faster than particles can escape

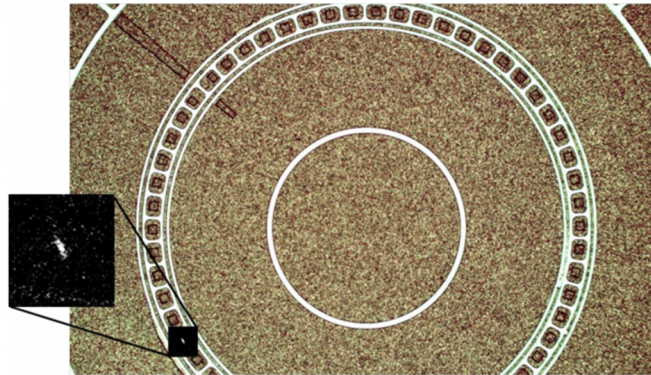
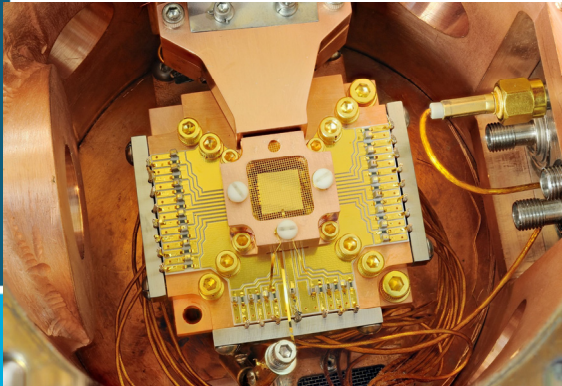
*Fortschr. Phys. 46 (1998) 4-5*

D. J. Wineland et al.: Experimental Primer ...

# Ion Traps



P. Schindler, et al., A quantum information processor with trapped ions, New J. Phys. 15 123012 (2013)



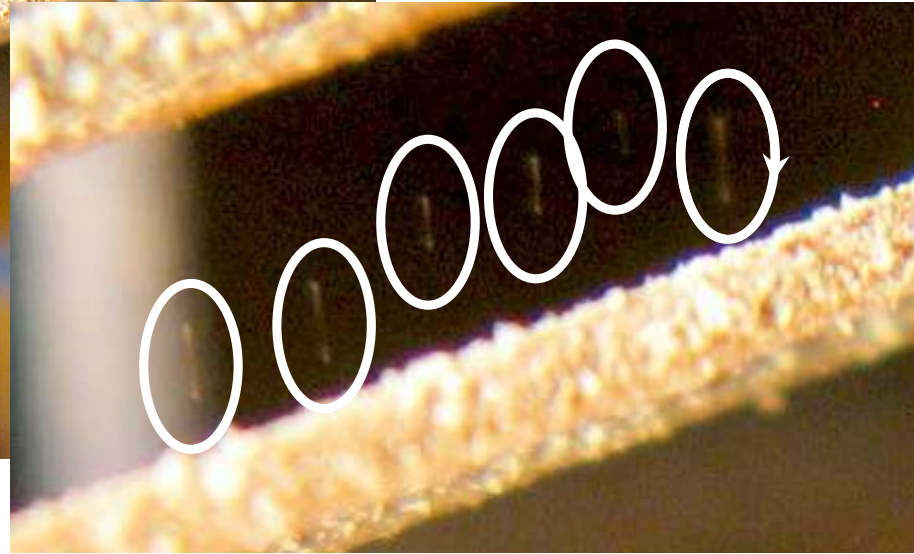
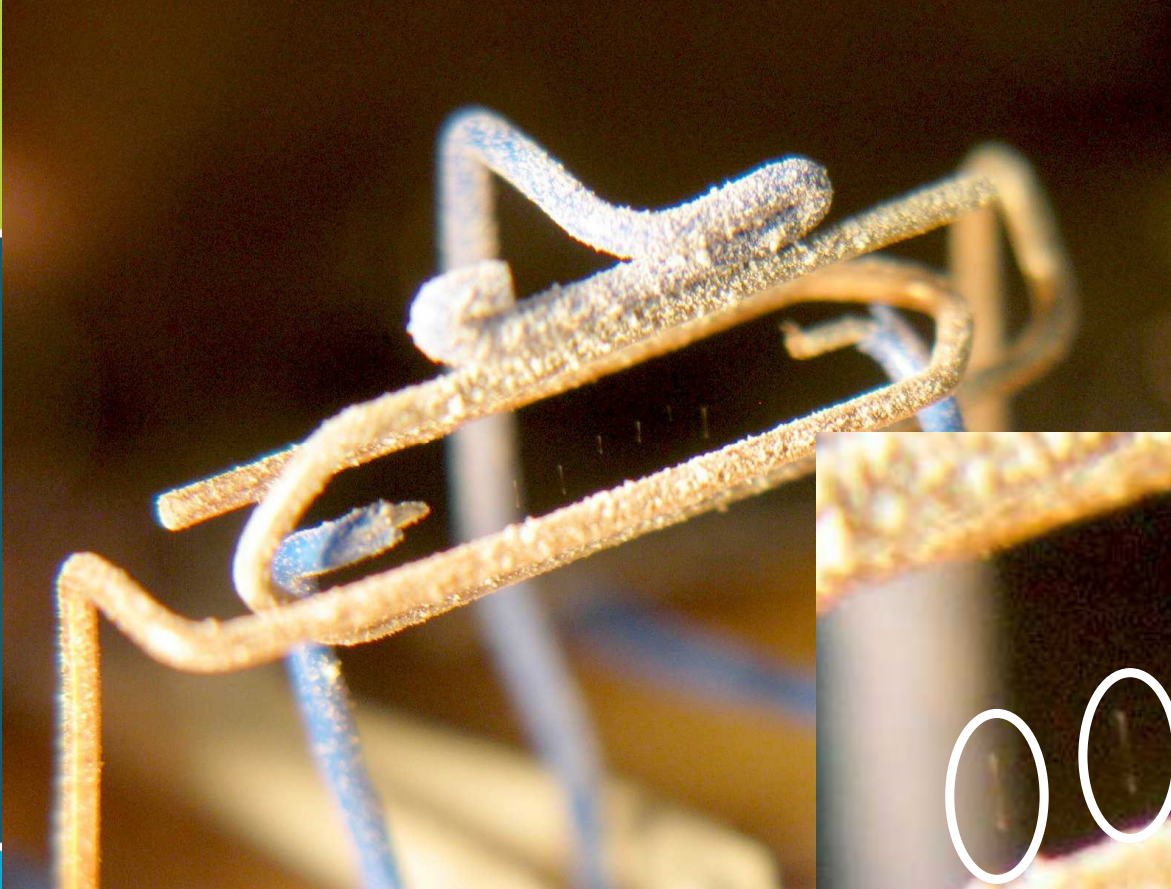
Microscope image of the ring trap surface, and image of an ion trapped above the surface. Microchip developed in collaboration between Univ. of Sussex and University of Nottingham

Chip ion trap for quantum computing from 2011 at NIST.

<https://www.sussex.ac.uk/physics/iqt/rsearch/ion-microchips/>

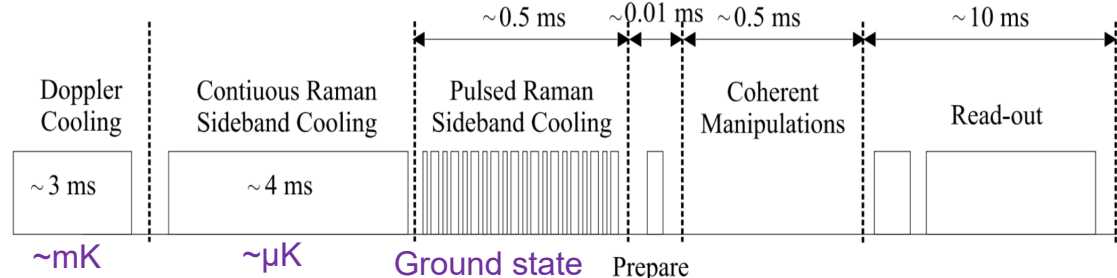
Even Paper Clip ion traps  
(well, dust particle traps)

Courtesy  
Tobias Schätz,  
LMU...Freiburg



# Challenges with Ion Traps

- Small number of ions/trap (~100's)
- Require very cold crystalline beams
  - Quantum phonon modes
  - Spatial Locality
- Each trap needs its own lasers
- Completely asynchronous
  - Characteristic times associated by decoherence and laser excitation and measurement times
- Error correction (requires many qubits)
- Entanglement limited to ions in single trap



An operation cycle for Quantum Computer

Entanglement of Two Trapped-Ion Spin Qubits, J. Home's PhD thesis, 2006

# Conditions for creating crystalline beams in storage rings



# Advantages of storage rings

- Can contain larger number of ions (>> thousands)
- Can contain large number ion chains
  - Each chain can be ~hundreds of ions
  - Each storage ring can contain ~thousands of chains
- Storage rings can be self-clocked (e.g., based on revolution freq.)
- Smaller set of lasers can operate on large number of ions.
  - Large degree of parallelism and multiplexing possible
- Having large number of ions and ion chains opens the way to new methods for error correction – even self-error correction
- Can employ multiple ion species – new opportunities

# Conditions for a Crystal : review

Two conditions are necessary to form and maintain crystalline beams

1. The storage ring must be alternating-gradient focusing and the energy must be less than the transition energy;  $g < g_{tr}$

This is required for stable motion under Coulomb interactions when particles are subject to bending in a storage ring. This condition is required for formation of a crystal.

2. The lattice periodicity is at least  $2\sqrt{2}$  greater than maximum betatron tune

This is to avoid linear resonances between phonon modes of the crystal and the lattice periodicity. This condition is required to maintain a crystal structure.

J. Wei, H. Okamoto, & A. Sessler, "Necessary Conditions for Attaining a Crystalline Beam," Phys. Rev. Let., V. 80, No. 12 (1998)

J. Wei, X. Li, & A. Sessler, "Crystalline beams," AIP Conf. Proc. 335,224 (1995) <https://aip.scitation.org/doi/abs/10.1063/1.48243>

X. Li, et al., "Phonon Modes and the Maintenance Condition of a Crystalline Beam," Proc. of 2005 PAC, Knoxville, TN

# Lattice with an imaginary $g_{tr}$ (or $-a$ ) ?

Azimuthal motion will be unbounded (from below) if  $\gamma \geq \gamma_{tr}$

The Hamiltonian can be expressed as<sup>1</sup>,

$$\tilde{H} = v_x J_x + v_y J_y + \frac{1-\gamma^2 F_z}{2} \tilde{P}_z^2 + \tilde{V}_C,$$

Where,  $\tilde{V}_C \approx \frac{k_z}{2} \bar{z}^2$ ,  $\bar{z} \ll \Delta_z$ ,  $k_z \geq 0$  and  $\langle F_z \rangle = \frac{\rho}{2\pi R} \oint_{Bend} D dt \equiv \frac{1}{\gamma_{tr}^2}$

If,

$\gamma > \gamma_{tr}$ ,  $1 - \gamma^2 F_z < 0$ , negative-mass regime, no ground state

$\gamma < \gamma_{tr}$ ,  $1 - \gamma^2 F_z > 0$ , motion is bounded (no negative energy states)

$F_z < 0$ ,  $1 - \gamma^2 F_z > 0$ , motion is bounded always

However, to maintain, lattice periodicity still needs to be high. So, a large  $\gamma_{tr}$  lattice would still need high periodicity/tune ratio.

# 1D crystal at very high energy

- Imaginary transition energy lattice

Jie Wei

BNL (USA) and IHEP (China)

H. Okamoto, H. Sugimoto

Hiroshima University (Japan)

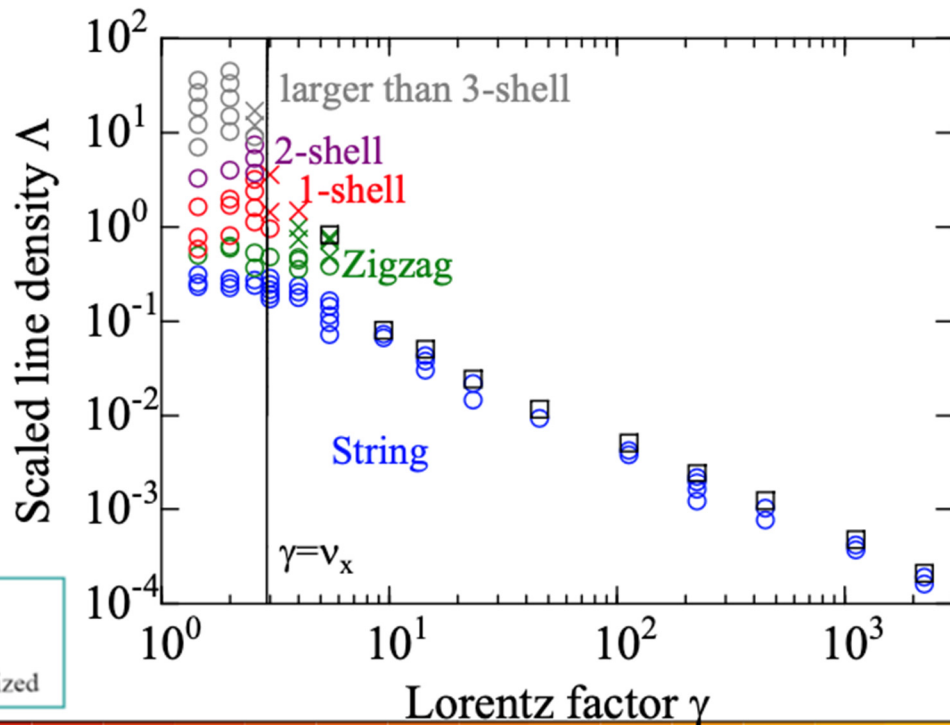
Y. Yuri

JAEA (Japan)

A.M. Sessler

LBNL (USA)

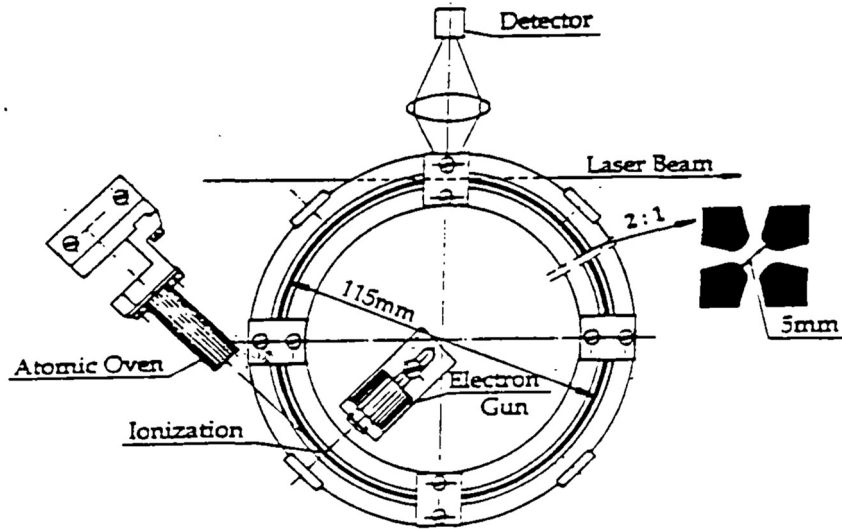
COOL'07 Workshop,  
September 10 - 14, 2007



# Circular Radio-Frequency Quadrupoles

# Circular RFQ Ion Traps

Example concept from A.G. Ruggiero



A. G. Ruggiero, *The Circular RFQ Storage Ring*, BNL Internal report, BNL-65920 (1998)

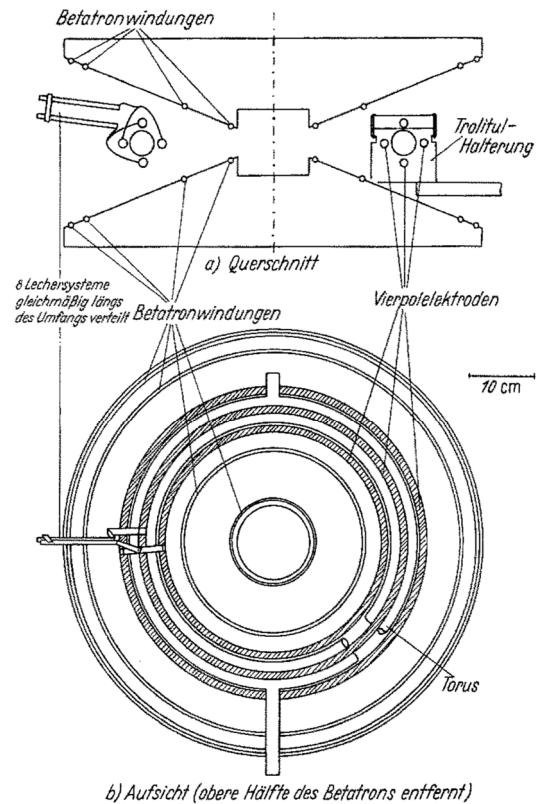


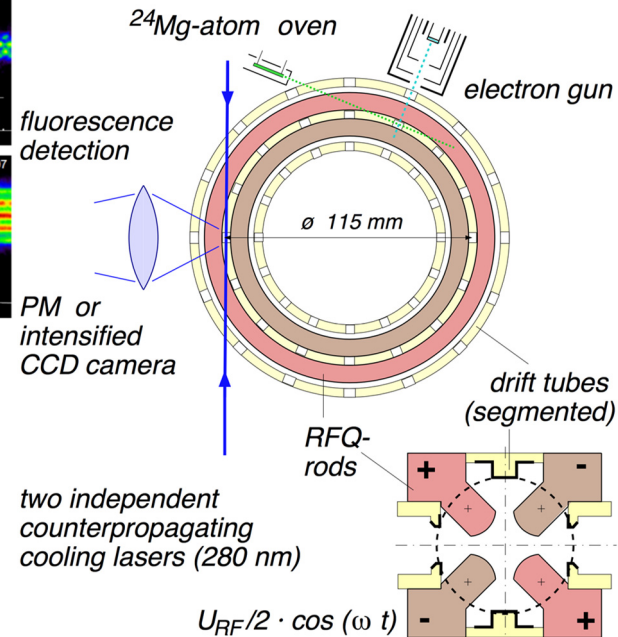
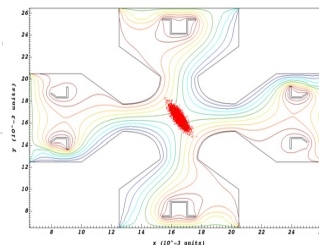
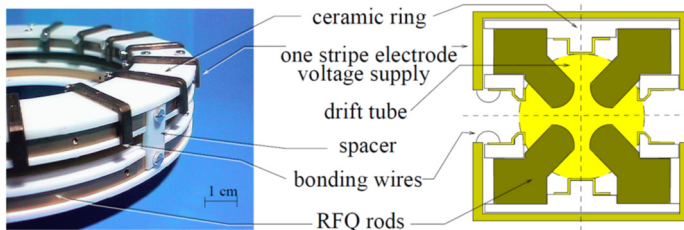
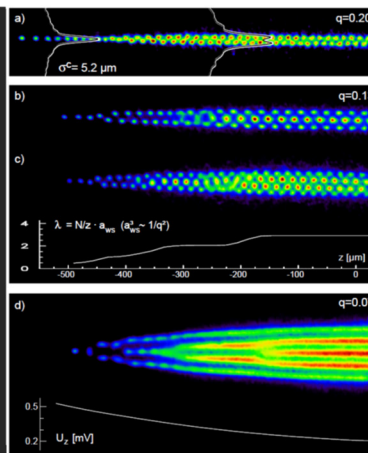
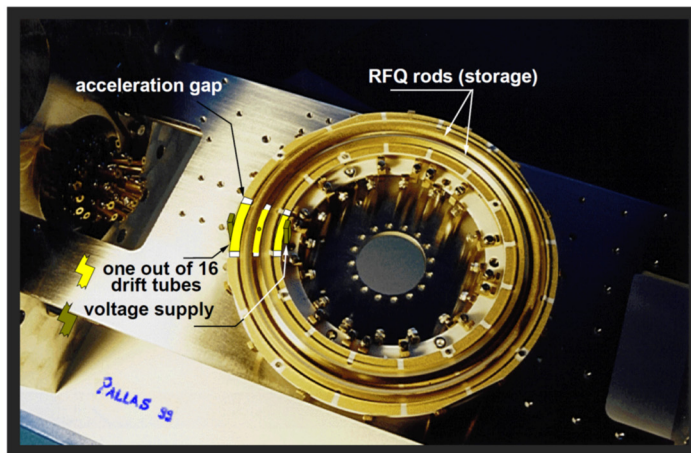
Fig. 1a u. b. Aufbau des Plasmabetatrons (schematisch)

J. Drees & W. Paul, *Zeitschrift für Physik* 180, 340-361 (1964)



# PALLAS RF-Quadrupole Ring Trap

U Schramm, T Schätz, M Bussmann and D Habs  
Sektion Physik, Ludwig-Maximilians-  
Universität München, D-85748 Garching,  
Germany



# Difference between AGS\* and CRFQ

CRFQ follows Mathieu's differential equation,

$$\frac{d^2 u}{d\xi^2} + [b_u - 2q_u \cos(2\xi)]u = 0, \text{ where } \xi = \frac{1}{2}\Omega t$$

AGS follows Hill's differential equation,

$$\frac{d^2 x}{ds^2} + K_x(s)x = 0, K_x \equiv \frac{B'}{B\rho} + \rho^{-2}, B' \equiv \partial B_y / \partial x$$

In an AGS, the beam traverses through FODO elements, experiencing sharp gradients at the entrances & exits to the magnets.

In a CRFQ, the RF traverses past the beam (an ion in a trap is stationary) but the gradients are gradual, following the sinusoidal rf potential.

\* Alternating Gradient Synchrotron

# A Storage Ring Ion Trap

For a circular RFQ, ion azimuthal motion is an angular precession at

$$\omega_0 = \beta c/R, \text{ where } R \text{ is radius of curvature of the ring.}$$

The alternating focusing period is

$$L = \beta \lambda, \text{ where } \lambda = \frac{c}{f_{rf}}, \text{ the rf field wavelength}$$

Periodicity, a function of velocity, is

$$P = \frac{C}{L}, \text{ where } C = 2\pi R \text{ and } L = \frac{v_0}{f_{rf}}$$

Electric potential for RFQ configuration is

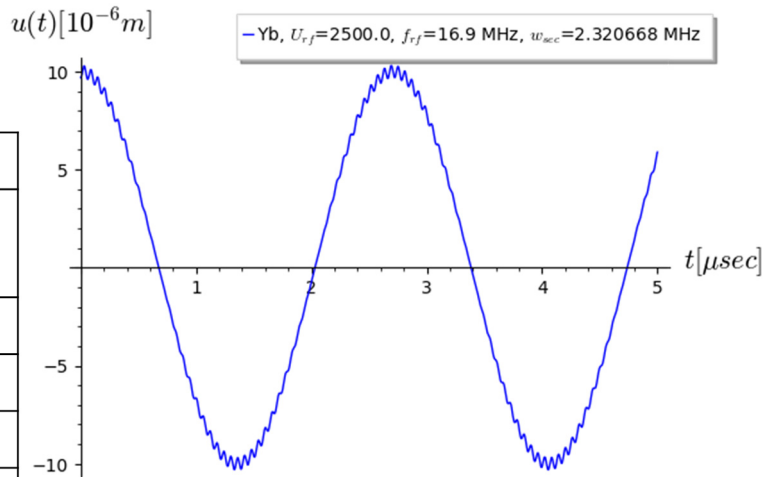
$$\phi(\hat{x}, \hat{y}, \hat{z}) = \frac{\phi_0(t)}{2r_0} (\hat{x}^2 - \hat{y}^2), \text{ where } \phi_0(t) = U_{dc} - U_{rf} \cos(\Omega t)$$

# A Storage Ring Ion Trap

Given this description and taking that  $\lambda_\beta \gg L^*$ , the storage ring can be described using very simple relations

Periodicity	$P = C/L$
Tune	$Q = \frac{C}{\lambda_\beta} = \frac{\omega_{sec} C}{2\pi v_0}$
Beta-function	$\beta_0 = R/Q$
phase	$\phi(z) = Qz/R$
dispersion	$D_0 = R/Q^2$
Momentum compaction	$\alpha = D_0/R = 1/Q^2$
Transition energy	$\gamma_{tr} = Q$

\*  $L = \frac{v_0}{f_{rf}}$ , the alternating focusing period



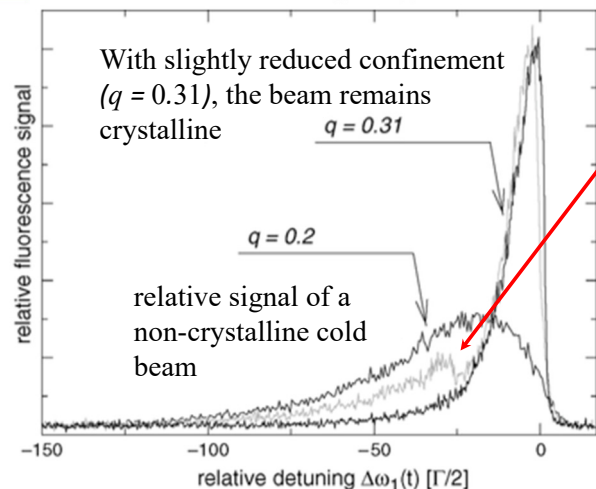
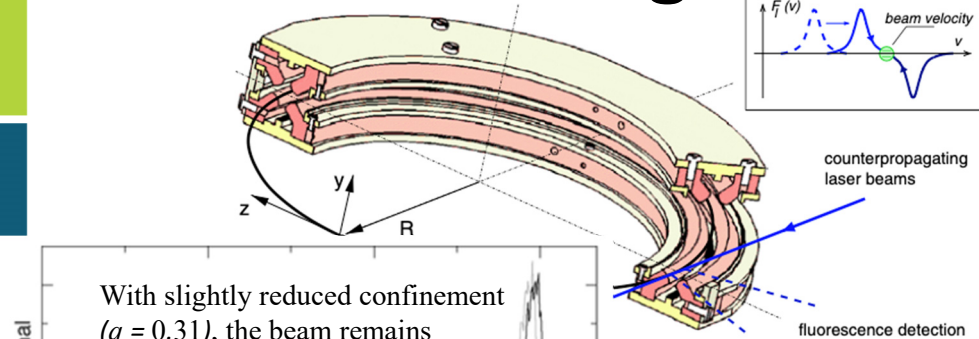
When  $b_u < q_u \ll 1$ , there is a simple solution to Mathieu's equation,

$$u(t) = u_{max} \left( 1 + \sqrt{2} \frac{\omega_{sec}}{\Omega} \cos \Omega t \right) \cos \omega_{sec} t$$

# Cooling

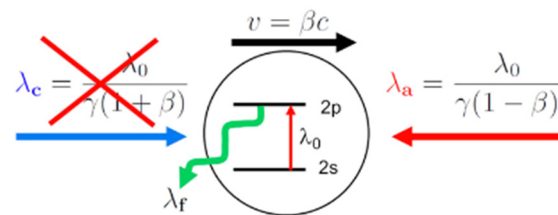
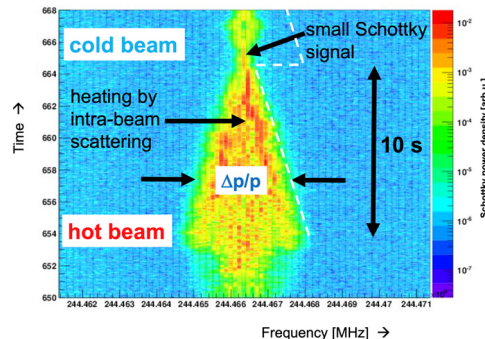


# Laser cooling

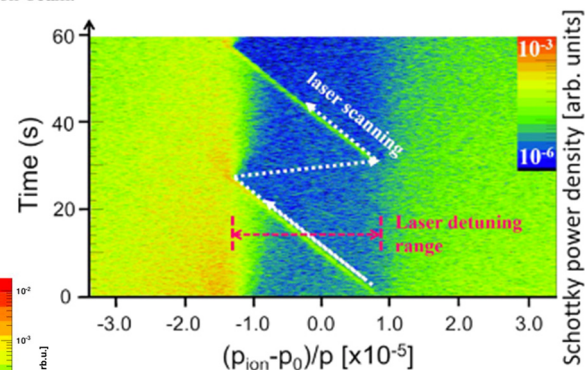


phase transition to the ordered beam ( $q = 0.33$ )

$C^{3+}$  ions stored in the ESR @ 122 MeV/u, scanning the wavelength of the cw laser



**Figure 1.** The principle of laser cooling: the ion absorbs a photon—thus momentum—from a counter-propagating laser beam, and deexcites by fluorescence. The counteracting force, required for laser cooling, is provided by the 'rf-bucket', which comes from bunching the ion beam.



*"The logarithmically color-coded Schottky power density shows that ions within the scanning range are decelerated to lower momenta by the laser force, leaving the revolution frequency range covered by the laser scan depopulated of ions"*

U. Schramm, D. Habs, Progress in Particle and Nuclear Physics 53 (2004) 583–677

Danyal Winters, et al., Laser cooling of relativistic heavy-ion beams for FAIR, Phys. Scr. T166 (2015) 014048 (6pp)



# 3D COOLING

- 1D Doppler Laser Cooling cannot cool below  $\sim 1$  mK.
  - Theoretical minimum is given by a balance between the dissipation and fluctuations. The doppler cooling limit is a result of the recoil energy averaged in all directions from the emitted photons and corresponds to  $kT_{min} = \frac{1}{2} \hbar \gamma$
- Sympathetic cooling has been done in ion traps. Different species of ion beams travelling together with the same velocity will exchange energy via Coulomb interaction. (micro-motion limits?)
- 3D cooling can be simulated using an electrostatic wiggler in the CRFQ storage ring lattice, with two counter-propagating lasers. (micro-motion limits?)
- Ultra-cold buffer gas cooling?
- Wild Idea = single ion detection cooling?

*Micromotion-Induced Limit to Atom-Ion Sympathetic Cooling in Paul Traps*

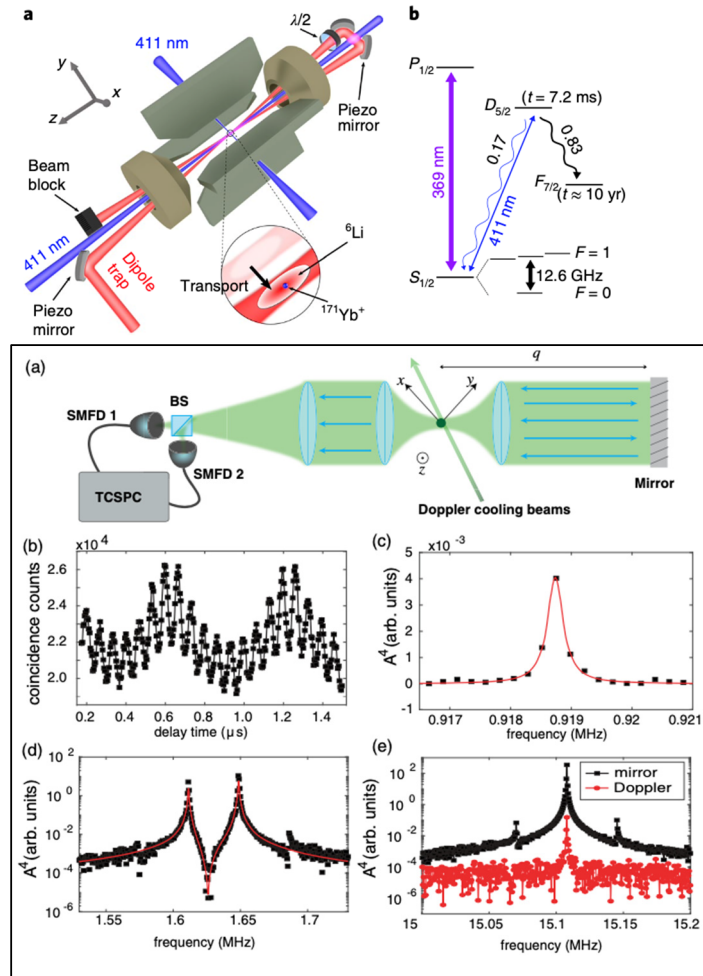
Marko Cetina, Andrew T. Grier, and Vlado Vuletic, PRL 109, 253201 (2012)

*Buffer gas cooling of a trapped ion to the quantum regime*

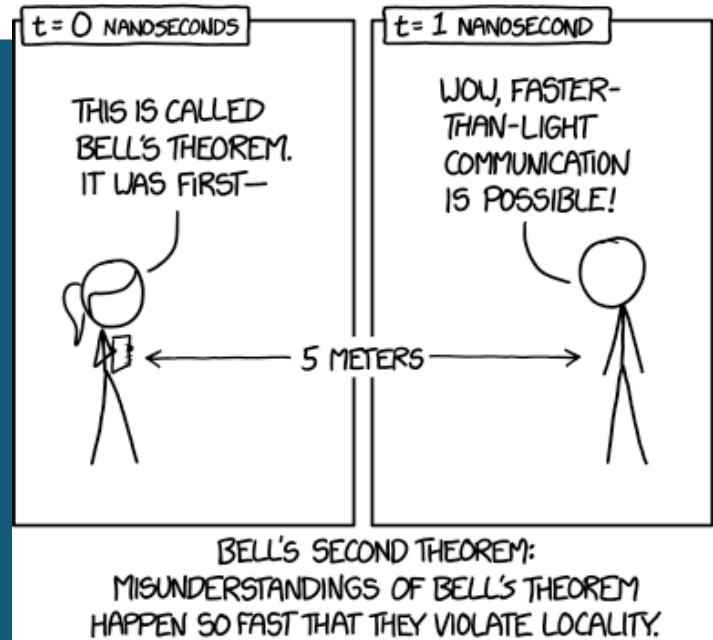
T. Feldker, H. Fürst, H. Hirzler, N. V. Ewald, M. Mazzanti, D. Wiater, M. Tomza and R. Gerritsma, Nature Physics Vol 16 April 2020 413–416

*Measuring Ion Oscillations at the Quantum Level with Fluorescence Light*

G. Cerchiari, G. Araneda, L. Podhora, L. Slodička, Y. Colombe, and R. Blatt  
Phys. Rev. Lett. 127, 063603 – Published 4 August 2021



# Entanglement



- Two independent wave functions interact
  - $|v\rangle = a_0|0\rangle + a_1|1\rangle$  and  $|w\rangle = b_0|\downarrow\rangle + b_1|\uparrow\rangle$
  - $|v\rangle \otimes |w\rangle = a_0b_0|0\rangle|\downarrow\rangle + a_1b_0|1\rangle|\downarrow\rangle + a_0b_1|0\rangle|\uparrow\rangle + a_1b_1|1\rangle|\uparrow\rangle$
  - $|v\rangle|w\rangle = r|0\rangle|\downarrow\rangle + s|1\rangle|\downarrow\rangle + t|0\rangle|\uparrow\rangle + u|1\rangle|\uparrow\rangle$
  - $r^2 + s^2 + t^2 + u^2 = 1$
  - If  $ru = st$ , states are not entangled
  - If  $ru \neq st$ , **states are entangled**
  - $\therefore (a_0b_0|0\downarrow\rangle)(a_1b_1|1\uparrow\rangle) \neq (a_0b_1|0\uparrow\rangle)(a_1b_0|1\downarrow\rangle)$ , are entangled (combined states are not a tensor product)
  - The wavefunctions can no longer be described independently even if moved far apart
- We must keep track of all eigenstates (accounting)
- To entangle internal spin states of two ions, we perform a gate operation using an external eigenstate (shared by the two ions)

Entanglement is the key to having fast algorithms in quantum computing. Changing the state of an entangled qubit will change the state of the paired qubit. Entanglement is necessary for a quantum algorithm to offer an exponential speed-up over a classical algorithm.

## Quantum C-NOT Gate

Combined wave function is,

$$|v\rangle|w\rangle = r|0\rangle|0\rangle + s|0\rangle|1\rangle + t|1\rangle|0\rangle + u|1\rangle|1\rangle$$

Where spin states are expressed as

$$|\downarrow\rangle = |0\rangle \text{ and } |\uparrow\rangle = |1\rangle. a_0b_0 = r, \text{ etc.} = \frac{1}{\sqrt{2}}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}}|1\rangle \\ |0\rangle \\ \frac{1}{\sqrt{2}}|1\rangle \\ |0\rangle \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} |1\rangle \\ |0\rangle \\ |0\rangle \\ |1\rangle \end{bmatrix} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

The original vector tensor product is unentangled.

The new vector outer and inner tensor products are not equal, so the wavefunctions are entangled.

In this case, eigenstates of two ions have become entangled.

# Entanglement in Ion traps

# Quantum Computations with Cold Trapped Ions

J. I. Cirac and P. Zoller\*

*Institut für Theoretische Physik, Universität Innsbruck, Technikerstrasse 25, A-6020 Innsbruck, Austria  
(Received 30 November 1994)*

A quantum computer can be implemented with cold ions confined in a linear trap and interacting with laser beams. Quantum gates involving any pair, triplet, or subset of ions can be realized by coupling the ions through the collective quantized motion. In this system decoherence is negligible, and the measurement (readout of the quantum register) can be carried out with a high efficiency.

PACS numbers: 89.80.+h, 03.65.Bz, 12.20.Fv, 32.80.Pj

## Entangled states of trapped atomic ions

Rainer Blatt<sup>1,2</sup> & David Wineland<sup>3</sup>

To process information using quantum-mechanical principles, the states of individual particles need to be entangled and manipulated. One way to do this is to use trapped, laser-cooled atomic ions. Attaining a general-purpose quantum computer is, however, a distant goal, but recent experiments show that just a few entangled trapped ions can be used to improve the precision of measurements. If the entanglement in such systems can be scaled up to larger numbers of ions, simulations that are intractable on a classical computer might become possible.

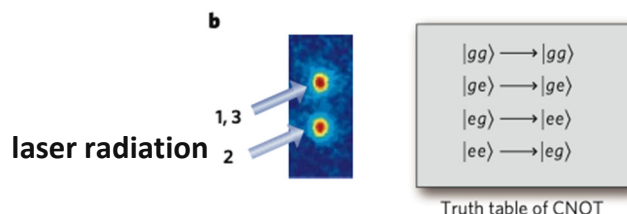
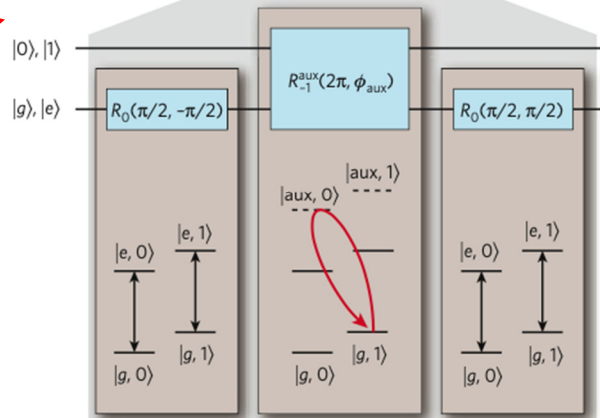
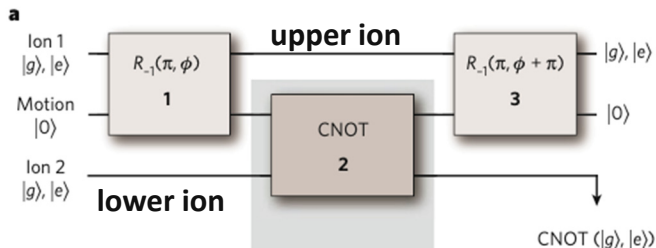
*Nature* **453**, 1008–1015 (2008).

<https://doi.org/10.1038/nature07125>

REVIEWS OF MODERN PHYSICS, VOLUME 93, APRIL–JUNE 2021

## Programmable quantum simulations of spin systems with trapped ions

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# Tackling Challenges

# Keeping track of thousands of ions

1. Need to locate – learn the equilibrium positions (EPs)
  1. Figuring them out is too complex<sup>1</sup> (although it only has to be done once, in principle = there are caveats)
  2. Mix ions of different isotopes to force dark area's and learn the pattern<sup>2</sup> (also problematic in a ring where the different ions follow different trajectories) – possible in linear ion traps.
  3. Don't worry about = learn and track on demand. Need precise clocks!
2. Need to index and uniquely identify, to allow precise operations
3. Need to know if an ion is lost – relearn the EP's and indices

$$H_0 = \sum_{\xi} \sum_{j=1}^N \frac{p_{\xi,j}^2}{2m} + V$$

Where  $V = V_{trap} + V_{Coulomb} = \frac{m}{2} \sum_{\xi} \sum_{j=1}^N (\omega_{\xi}^2 \xi_j^2) + \sum_{n,j} \frac{e^2}{4\pi\epsilon_0 r_{n,j}}$

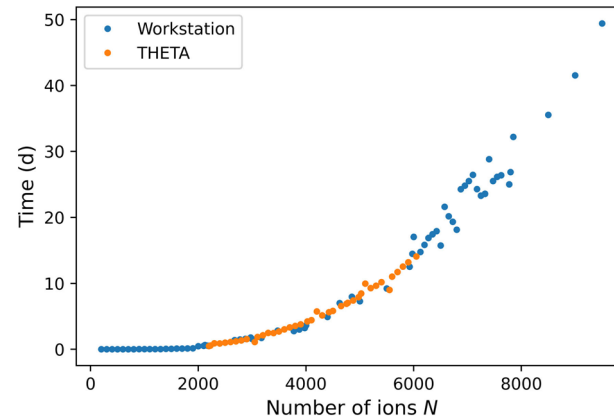
Equilibrium positions are determined by

$$\left( \frac{\partial V}{\partial \xi_i} \right)_{\xi_i = \xi_{i,0}} = 0, \quad (i = 1, 2, \dots, N)$$

$\xi_{i,0}$  is the equilibrium position ( $x_i, y_i, z_i = 0$ ) for ion  $i$

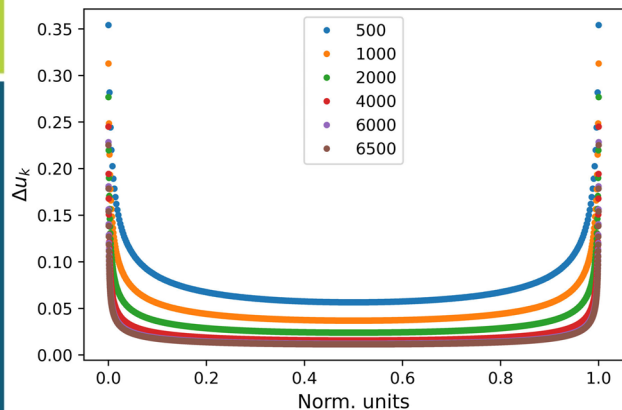
Define the dimensionless parameter  $u_i \equiv \xi_{i,0}/z_s$

Then, for ion  $j$ ,  $u_j - \sum_{n=1}^{j-1} \frac{1}{(u_j - u_n)^2} + \sum_{n=j+1}^N \frac{1}{(u_j - u_n)^2} = 0$



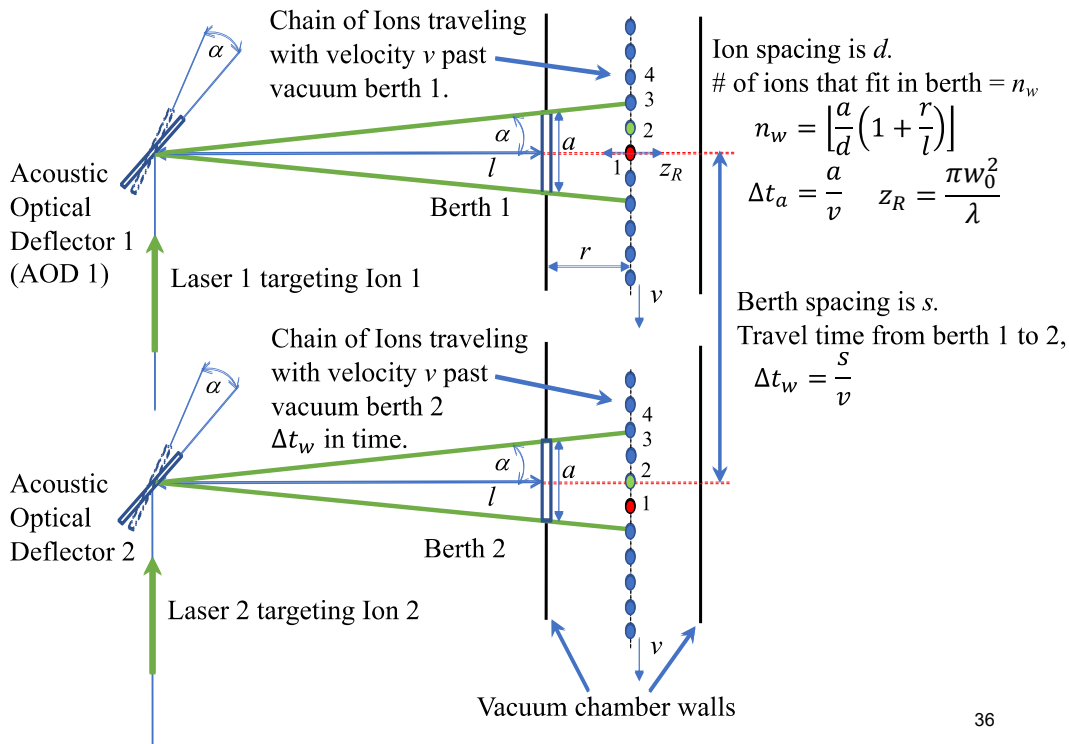
Computation time as function of the number of ions  $N$ , in days. The computation time grows as  $\sim N^3$ .

# Don't worry about it?



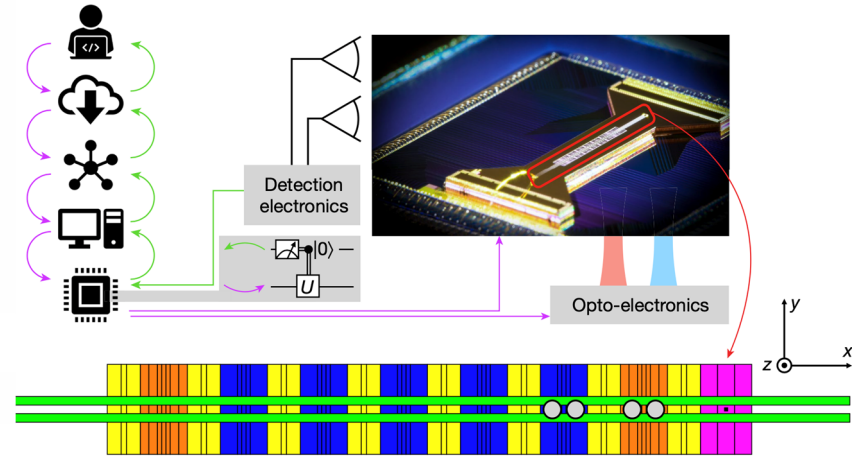
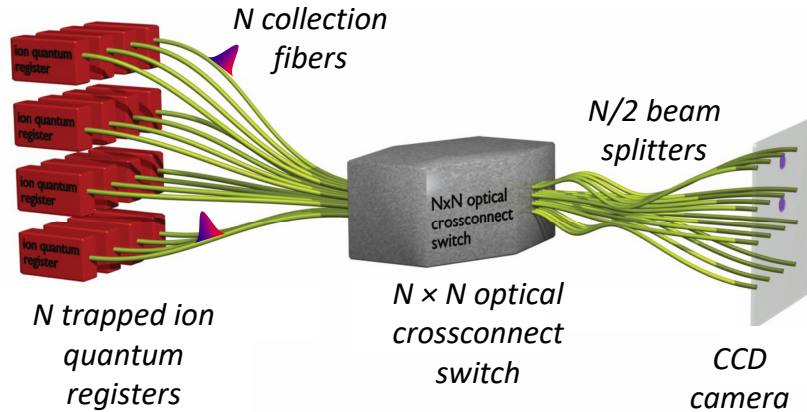
The distance between adjacent ions in the ICC for different values of  $N$ . Note that  $\Delta u_k$  converges to a finite distance as  $N$  becomes larger.

As you find them, you can quickly learn them and deduce  $d$ 's.





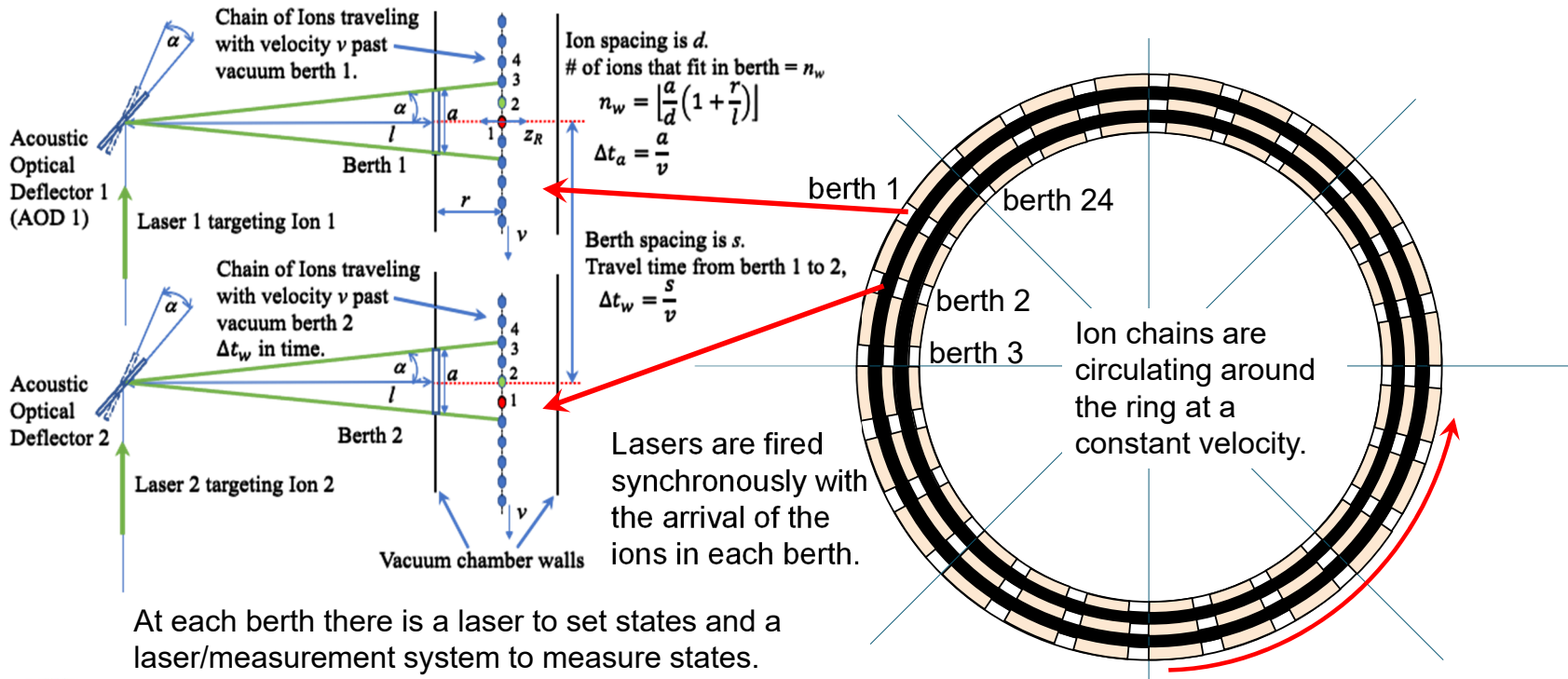
# (to 1,000,000 qubits?)



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Conceptual Example, for a circular accelerator. Other configurations are allowed; racetrack (2 straight sections), triangular (3 straight sections), rectangular/square (4 straight sections), etc. Here there are 24 berths (or physical windows in a vacuum chamber in this case). Laser operations will take place at each berth, allowing many operations to be happening simultaneously.



Note: just schematic, nothing is to scale.

# Summary +

- Storage rings for QIS are a viable and perhaps, even, a necessary technology, with the ability to scale to large numbers of qubits, but also offering improved error correction (which I didn't talk about but is an important topic)
- QIS with relativistic beams is a new topic. Since we are dealing with quantum systems, the process involves setting a state and then making a measurement. The quantum states will be destroyed in the process. Other quantum properties can be used as qubits.
- There are many interesting problems to work on. We encourage folks to get involved and dive into the problem that interest you.

# To Learn More, see

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