Analytical theory of McMillan map
WEA1CO06

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This presentation is dedicated to Dr. Slava Danilov

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for their valuable comments and discussions on this topic.

1.1 Between **Scylla**[1] and **Charybdis**[2]

[1]**Scylla** (Greek: $\Sigma\kappa\nu\lambda\lambda\alpha$) — was a monster representing fractal phase space and threatening the dynamic aperture
[2]**Charybdis** (Greek: $X\alpha\rho\nu\beta\delta\iota\sigma$) — was a sea monster (whirlpool) representing collective instability

1.2 Accelerators, Maps & Puzzles

- **Repetitive nature**

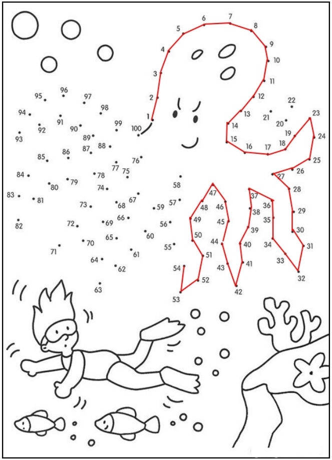
$$\begin{aligned} q' &= q'(q,p) \\ p' &= p'(q,p) \end{aligned}$$

- **Symplectic structure**

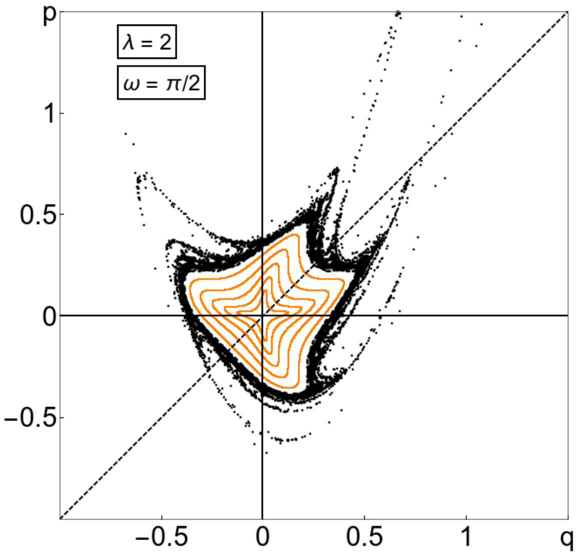
$$\frac{\partial q'}{\partial q}\frac{\partial p'}{\partial p} - \frac{\partial q'}{\partial p}\frac{\partial p'}{\partial q} = 1$$

- **Invariant curves (if any)**

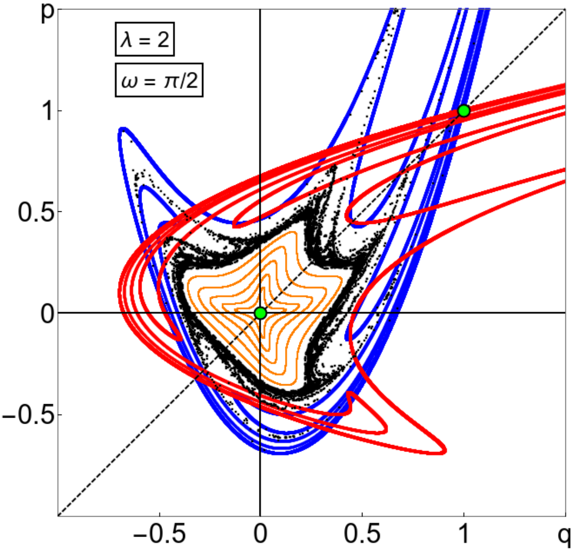
$$\mathcal{K}(q,p) = \mathcal{K}(q',p')$$



1.3 Hénon map: $\begin{bmatrix} q' \\ p' \end{bmatrix} = \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} q \\ p - \lambda\, q^2 \end{bmatrix}$



1.3 Hénon map: $\begin{bmatrix} q' \\ p' \end{bmatrix} = \begin{bmatrix} -p + \lambda\, q^2 \\ q \end{bmatrix}$



2.1 McMillan map

Canonical McMillan map

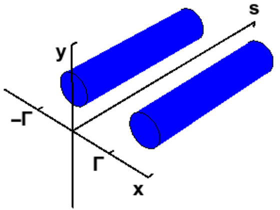
$$M_{\mathrm{c}}: \qquad q' = -p - \frac{2\,\epsilon\,q}{q^2+\Gamma}$$
$$p' = q$$

Canonical biquadratic

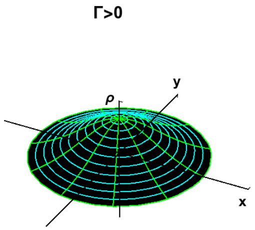
$$\mathcal{B}_{\mathrm{c}}(q,p,\mathcal{K}) \equiv q^2 p^2 + \Gamma\left(q^2+p^2\right) + 2\,\epsilon\,q\,p + \mathcal{K}$$
$$= \begin{bmatrix} q^2 \\ q \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \Gamma \\ 0 & 2\,\epsilon & 0 \\ \Gamma & 0 & \mathcal{K} \end{bmatrix} \begin{bmatrix} p^2 \\ p \\ 1 \end{bmatrix} = 0$$

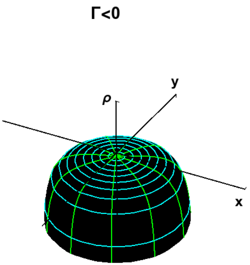
2D optics

2D magnetostatic lens (unstable?)

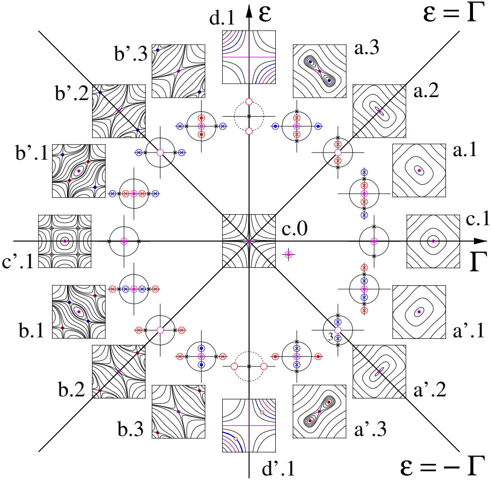


Electron lens



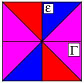


2.2 Bifurcation diagram for canonical McMillan map



Fixed points and 2−cycles	center	saddle	neutral
$f_1 = 0$	⊙	⊗	○
$f_{2,3} = \pm\sqrt{-\varepsilon-\Gamma}$	⊙	⊗	○
$c_{1,2} = \pm\sqrt{\varepsilon-\Gamma}$	⊙	⊗	○

Stability



Critical points of K(q,p)	extrema	saddle
$K = 0$	•	⊹
$K = (\varepsilon+\Gamma)^2$	•	⊹
$K = (\varepsilon-\Gamma)^2$	•	⊹
$K = \Gamma^2$		⊕

Elliptic integrals and Jacobi elliptic functions

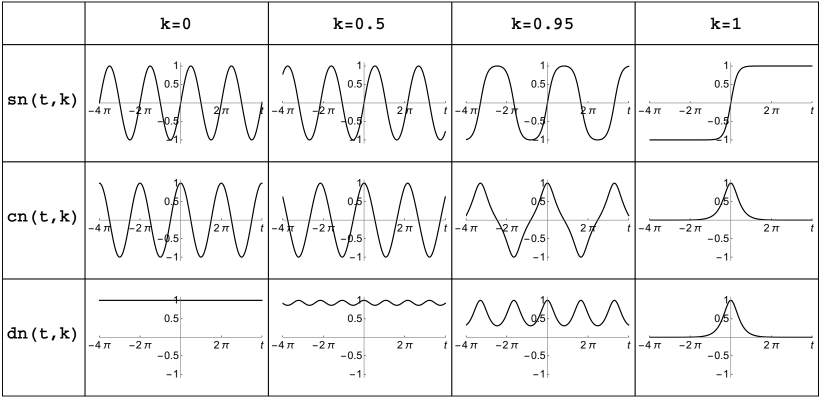
$$
\begin{aligned}
\mathrm{F}(t,k) &= \int_0^t \frac{\mathrm{d}\,\tau}{\sqrt{(1-\tau^2)(1-k^2\,\tau^2)}} \\
\mathrm{E}(t,k) &= \int_0^t \sqrt{\frac{k'^2+k^2\,\tau^2}{1-\tau^2}}\,\mathrm{d}\,\tau \\
\Pi(t,\alpha^2,k) &= \int_0^t \frac{\mathrm{d}\,\tau}{(1-\alpha^2\tau^2)\sqrt{(1-\tau^2)(\tau^2-k'^2)}}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{sn}^{-1}(t,k) &= \int_0^t \frac{\mathrm{d}\,\tau}{\sqrt{(1-\tau^2)(1-k^2\,\tau^2)}} \\
\mathrm{cn}^{-1}(t,k) &= \int_t^1 \frac{\mathrm{d}\,\tau}{\sqrt{(1-\tau^2)(k'^2+k^2\,\tau^2)}} \\
\mathrm{dn}^{-1}(t,k) &= \int_t^1 \frac{\mathrm{d}\,\tau}{\sqrt{(1-\tau^2)(\tau^2-k'^2)}}
\end{aligned}
$$

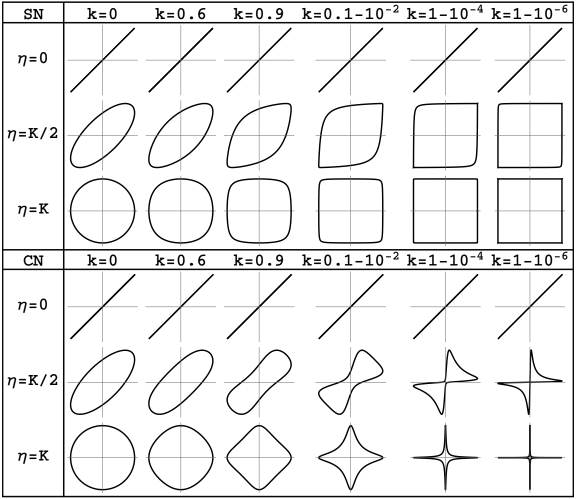
$$k' = \sqrt{1-k^2}$$

Plot of Jacobi elliptic functions

$$\mathrm{sn}^2 t + \mathrm{cn}^2 t = 1 \qquad\qquad k^2 \mathrm{sn}^2 t + \mathrm{dn}^2 t = 1$$



Elliptic Lissajous curves for SN and CN ($m = n = 1$)



2.3 Normal forms (A. Iatrou and J. Roberts)

$q, p \to q, p/\sqrt[4]{|\mathcal{K}|}$ $\epsilon, \Gamma \to \epsilon, \Gamma/\sqrt{|\mathcal{K}|}$

$$q^2 p^2 + \Gamma(q^2+p^2) + 2\epsilon\, q\, p \pm 1 = 0, \;\text{ for } \mathcal{K} \gtrless 0$$

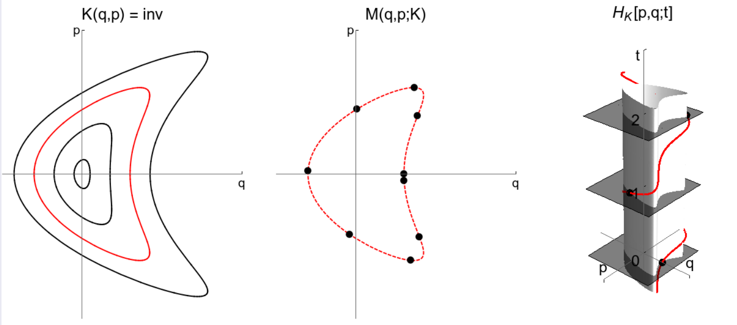
$$q_n = q(n) \qquad q(t) = \sqrt[4]{|\mathcal{K}|}\, f(k,k')\, \mathrm{ef}\,(u_0 + \eta\, t, k)$$

$$p_n = p(n) \qquad p(t) = \sqrt[4]{|\mathcal{K}|}\, f(k,k')\, \mathrm{ef}\,(u_0 + \eta\,(t+1), k)$$

	a.1, a'.1	a.3, a'.3	a.3, a'.3	b.1, b'.1
$\mathcal{K} \in$	$(-\infty; 0] \le 0$	$(-\infty; 0] \le 0$	$\left[0; (\lvert\epsilon\rvert - \Gamma)^2\right] \ge 0$	$\left[0; (\lvert\epsilon\rvert + \Gamma)^2\right] \ge 0$
B	$\frac{k}{k'} - \frac{k'}{k} \in \mathbb{R}$	$\frac{k}{k'} - \frac{k'}{k} \in \mathbb{R}$	$\frac{1}{k'} + k' \ge 2$	$\frac{1}{k} + k \ge 2$
$k(B)$	$\frac{1}{\sqrt{2}}\sqrt{1+\frac{B}{\sqrt{B^2+4}}}$	$\frac{1}{\sqrt{2}}\sqrt{1+\frac{B}{\sqrt{B^2+4}}}$	$\frac{\sqrt[4]{B^2-4}\sqrt{B-\sqrt{B^2-4}}}{\sqrt{2}}$	$\frac{B-\sqrt{B^2-4}}{2}$
η	$\mathrm{arcds}\sqrt{\frac{k\,k'\,\Gamma}{\sqrt{\lvert\mathcal{K}\rvert}}}$	$\mathrm{arcds}\sqrt{\frac{k\,k'\,\Gamma}{\sqrt{\lvert\mathcal{K}\rvert}}}$	$\mathrm{arccs}\sqrt{\frac{k'\,\Gamma}{\sqrt{\mathcal{K}}}}$	$\mathrm{arcns}\sqrt{\frac{-k\,\Gamma}{\sqrt{\mathcal{K}}}}$
u_0	$\mathrm{arccn}\,\frac{q_0}{\sqrt[4]{\lvert\mathcal{K}\rvert}}\sqrt{\frac{k'}{k}}$	$\mathrm{arccn}\,\frac{q_0}{\sqrt[4]{\lvert\mathcal{K}\rvert}}\sqrt{\frac{k'}{k}}$	$\mathrm{arcdn}\,\frac{q_0\sqrt{k'}}{\sqrt[4]{\mathcal{K}}}$	$\mathrm{arcsn}\,\frac{q_0}{\sqrt{k}\sqrt[4]{\mathcal{K}}}$
$q/\sqrt[4]{\lvert\mathcal{K}\rvert}$	$\sqrt{\frac{k}{k'}}\,\mathrm{cn}\,[u_0 - n\,\eta]$	$\sqrt{\frac{k}{k'}}\,\mathrm{cn}\,[u_0 - n\,\eta]$	$\sqrt{\frac{1}{k'}}\,\mathrm{dn}\,[u_0 - n\,\eta]$	$\sqrt{k}\,\mathrm{sn}\,[u_0 - n\,\eta]$
$p/\sqrt[4]{\lvert\mathcal{K}\rvert}$	$\sqrt{\frac{k}{k'}}\,\mathrm{cn}\,[u_0 - (n+1)\eta]$	$\sqrt{\frac{k}{k'}}\,\mathrm{cn}\,[u_0 - (n+1)\eta]$	$\sqrt{\frac{1}{k'}}\,\mathrm{dn}\,[u_0 - (n+1)\eta]$	$\sqrt{k}\,\mathrm{sn}\,[u_0 - (n+1)\eta]$

3.1 The discrete Theorem of Liouville

If a symplectic map $\Phi : \mathbb{F}^{2n} \to \mathbb{F}^{2n}$ has n independent integrals in involution $\mathcal{K}_1, \ldots, \mathcal{K}_n$, then any compact non-singular level M is a disconnected union of tori, on which Φ defines a collection of shifts. The angle variables are constructed in the same way as in the usual Liouville theorem related to the family of integrable Hamiltonian systems with Hamiltonians $\mathcal{K}_1, \ldots, \mathcal{K}_n$.



3.2 Action variable

$$J = \frac{1}{2\pi}\oint p\,\mathrm{d}q = \frac{\sqrt{|\mathcal{K}|}}{2\pi}\oint \tilde{p}\,\mathrm{d}\tilde{q} = \frac{\sqrt{|\mathcal{K}|}}{2\pi} \times \begin{cases} k \;\; S_{\mathrm{sn}} \\ \frac{k}{k'}\, S_{\mathrm{cn}} \\ \frac{1}{k'}\, S_{\mathrm{dn}} \end{cases}$$

where $S_{\mathrm{ef}} = \oint \mathrm{ef}(t+\eta;k)\,\mathbf{d}\,\mathrm{ef}(t;k)$ is the area of corresponding elliptic Lissajous curve with commensurate frequencies, $m/n = 1$.

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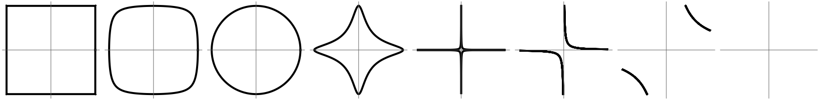
where $S_{\mathrm{ef}} = \oint \mathrm{ef}(t+\eta;k)\,\mathbf{d}\,\mathrm{ef}(t;k)$ is the area of corresponding elliptic Lissajous curve with commensurate frequencies, $m/n = 1$.

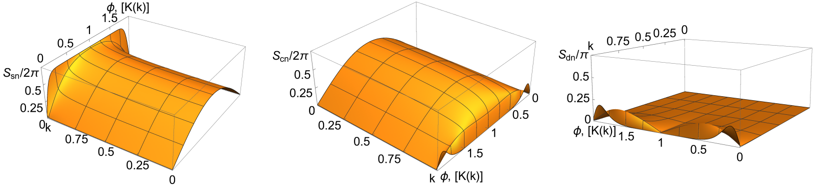
$$S_{\mathrm{sn}} = \boxed{\frac{4}{k^2}\frac{1}{\mathrm{sn}^3\eta}\left[-\mathrm{dn}^2\eta\,\mathrm{K} + \mathrm{sn}^2\eta\,\mathrm{E} + \mathrm{cn}^2\eta\,\mathrm{dn}^2\eta\,\Pi(k^2\mathrm{sn}^2\eta,k)\right]}$$

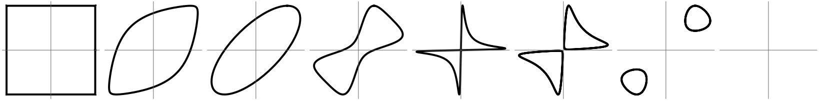
$$S_{\mathrm{cn}} = \boxed{\frac{4}{k^2}\frac{\mathrm{dn}\eta}{\mathrm{sn}^3\eta}\left[\mathrm{K} - \mathrm{sn}^2\eta\,\mathrm{E} - \mathrm{cn}^2\eta\,\Pi(k^2\mathrm{sn}^2\eta,k)\right]}$$

$$S_{\mathrm{dn}} = \boxed{2\,\frac{\mathrm{cn}\eta}{\mathrm{sn}^3\eta}\left[(\mathrm{sn}^2\eta+\mathrm{dn}^2\eta)\mathrm{K} - \mathrm{sn}^2\eta\,\mathrm{E} - \mathrm{dn}^2\eta\,\Pi(k^2\mathrm{sn}^2\eta,k)\right]}$$

3.3 $S_{\rm sn}$, $S_{\rm cn}$, $S_{\rm dn}$







4.1 Angle variable

Fourier series

$$q(t) = \sqrt[4]{|\mathcal{K}|}\, f(k,k')\, \mathrm{ef}\,(u_0 + \eta\, t, k) \qquad q_n = q(n)$$

$$\mathrm{sn}\,\eta = \frac{\pi}{k\,\mathrm{K}} \sum_{n=0}^{\infty} \mathrm{csch}\left[(2n+1)\,\frac{\pi\,\mathrm{K}'}{2\,\mathrm{K}}\right] \sin\left[2\pi(2n+1)\,\boxed{\frac{\eta}{4\,\mathrm{K}}}\,\right]$$

$$\mathrm{cn}\,\eta = \frac{\pi}{k\,\mathrm{K}} \sum_{n=0}^{\infty} \mathrm{sech}\left[(2n+1)\,\frac{\pi\,\mathrm{K}'}{2\,\mathrm{K}}\right] \cos\left[2\pi(2n+1)\,\boxed{\frac{\eta}{4\,\mathrm{K}}}\,\right]$$

$$\mathrm{dn}\,\eta = \frac{\pi}{2\mathrm{K}} + \frac{\pi}{\mathrm{K}} \sum_{n=1}^{\infty} \mathrm{sech}\left[\frac{n\,\pi\,\mathrm{K}'}{\mathrm{K}}\right] \cos\left[2\pi\, n\,\boxed{\frac{\eta}{2\,\mathrm{K}}}\,\right]$$

Let $\mathcal{D} : \mathbb{R}^2 \to \mathbb{R}^2$ be the area-preserving integrable map which has an invariant of motion $\mathcal{K}(q,p) = \mathcal{K}(q',p')$ where (q',p') is an image of (q,p), then frequency for every trajectory can be calculated as:

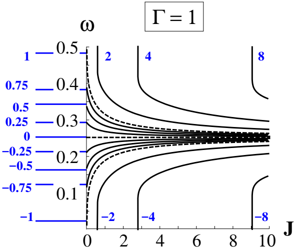
$$\nu = \frac{\int_q^{q'} \left(\frac{\partial \mathcal{K}}{\partial p}\right)^{-1} \mathrm{d}\, q}{\oint \left(\frac{\partial \mathcal{K}}{\partial p}\right)^{-1} \mathrm{d}\, q}.$$

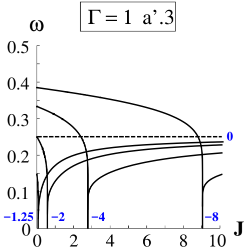
Proof: □

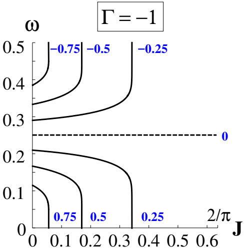
$$\nu = \frac{1}{T} = \frac{\int_q^{q'} \frac{\mathrm{d}t}{\mathrm{d}q}\,\mathrm{d}q}{\oint \frac{\mathrm{d}t}{\mathrm{d}q}\,\mathrm{d}q} = \frac{\int_q^{q'} \left(\frac{\partial H}{\partial p}\right)^{-1} \mathrm{d}q}{\oint \left(\frac{\partial H}{\partial p}\right)^{-1} \mathrm{d}q} = \frac{\int_q^{q'} \cancel{\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}H}} \left(\frac{\partial \mathcal{K}}{\partial p}\right)^{-1} \mathrm{d}q}{\oint \cancel{\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}H}} \left(\frac{\partial \mathcal{K}}{\partial p}\right)^{-1} \mathrm{d}q}$$

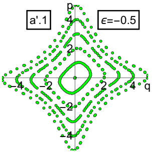
Q.E.D. ■

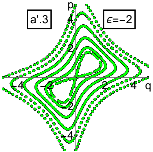
4.3 Frequency-amplitude dependence, $|\Gamma| = 1$

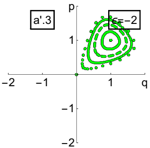


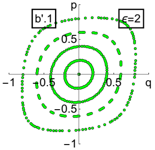












Summary

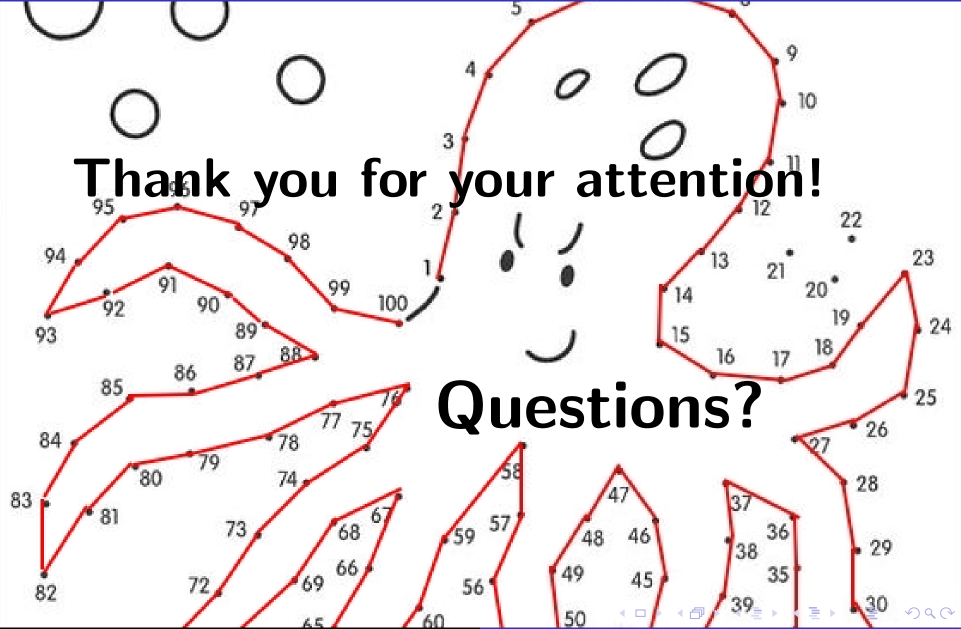
- Application of most general 1D asymmetric McMillan map for accelerator physics (1- and 2-cell lattices)

- Solutions for all finite trajectories of canonical McMillan map via Jacobi elliptic functions; normal forms

- Analytical expressions for mechanical analogies of action-angle variables

- Description of horizontal and vertical planes of 2D magnetostatic McMillan lens and any plane with zero total angular momentum for axially symmetric electron McMillan lens

- Proof and check of *Danilov Theorem*



Thank you for your attention!

Questions?