Fokker-Planck analysis of transverse collective instabilities in electron storage rings

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- Emission of synchrotron radiation affects the instability threshold

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- First analysis of single bunch instabilities using the Fokker-Planck equation was made by T. Suzuki[†], who focused on the zero-chromaticity limit and the traditional transverse mode coupling instability (TMCI)

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  - Found that the Fokker-Planck dynamics implies that higher-order modes are damped more strongly
  - Since TMCI describes the merger of two low-order modes, the Fokker-Planck analysis makes a relatively small effect on the predicted instability threshold when $\xi = 0$
  - At large chromaticity we find that stability is dictated by high order modes, and the damping and diffusion of the Fokker-Planck equation increases the predicted stable current

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- We have simplified Suzuki's results, and applied them to "large" chromaticity

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Overview of the Fokker-Planck analysis

$$\frac{\partial F}{\partial s} + \{F, \mathcal{H}\} = \frac{2}{c\tau_z}\left[\sigma_\delta^2 \frac{\partial^2 F}{\partial p_z^2} + p_z \frac{\partial F}{\partial p_z} + F\right] + \frac{2}{c\tau_x}\left[\varepsilon_0 \mathcal{J} \frac{\partial^2 F}{\partial \mathcal{J}^2} + \frac{\varepsilon_0}{4\mathcal{J}} \frac{\partial^2 F}{\partial \Psi^2} + (\varepsilon_0 + \mathcal{J}) \frac{\partial F}{\partial \mathcal{J}} + F\right]$$

Overview of the Fokker-Planck analysis

Hamiltonian part:
linear (synchrotron + betatron) motion,
chromatic nonlinearity, and transverse wakefields

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Longitudinal damping time

Energy spread

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Longitudinal damping time Energy spread Transverse damping time Natural emittance

1. Linearize for perturbations about equilibrium

$$\underbrace{F(z, p_z, \Psi, \mathcal{J}; s)}_{\text{Distribution function}} = \underbrace{f_0(\mathcal{J}) g_0(\mathcal{H}_z)}_{\text{Equilibrium}} + \underbrace{f_1(\Psi, \mathcal{J}; s) g_1(z, p_z; s)}_{\text{Perturbation}}$$

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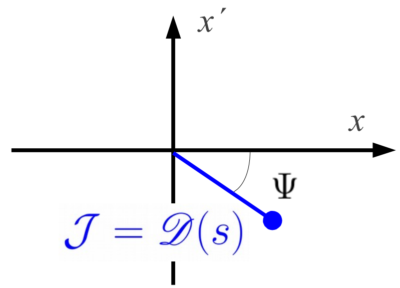
2. Assume that the transverse motion is described by dipole oscillations† at the (chromaticity-corrected) betatron frequency



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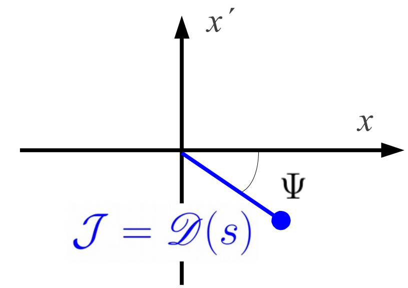

3. Expand longitudinal perturbation as a sum of linear modes in longitudinal action and angle

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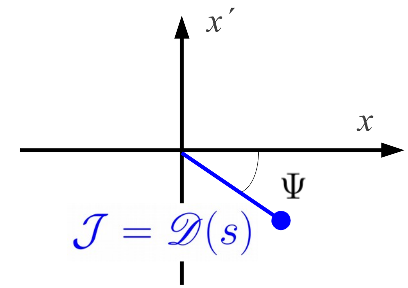

3. Expand longitudinal perturbation as a sum of linear modes in longitudinal action and angle

4. Solve eigenvalue problem to determine normal modes and complex frequencies as a function of current and chromaticity

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Linearized Fokker-Planck equation

- Assumptions for distribution function imply

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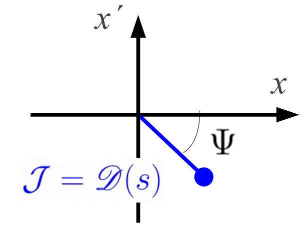

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Transverse betatron oscillation
in action-angle coordinates $(\mathcal{J},\Psi)$

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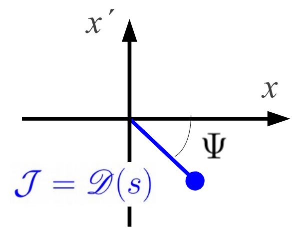

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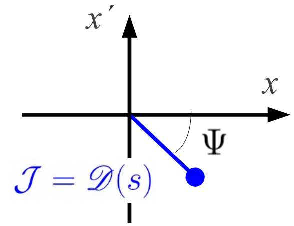

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Longitudinal perturbation

Transverse betatron oscillation in action-angle coordinates $(\mathcal{J},\Psi)$

Complex mode frequency

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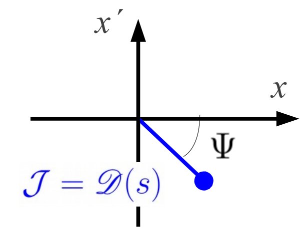

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Complex mode frequency

Head-tail phase, $k_\xi \equiv \dfrac{\omega_0 \xi_x}{\alpha_c c}$

- Linearized Fokker-Planck equation for longitudinal perturbation g_1 becomes

$$\frac{\Omega + i/\tau_x}{c} g_1(z, p_z) + i\{g_1, \mathcal{H}_z\} - \overbrace{\boxed{\frac{2\pi I g_0(\mathcal{I})}{\gamma c I_A Z_0} \int d\hat{p}_z d\hat{z}\ \beta_x W_D(z - \hat{z}) e^{ik_\xi(\hat{z} - z)} g_1(\hat{z}, \hat{p}_z)}}^{\text{Contribution of dipolar wakefield}} = \frac{2i}{c\tau_z}\left[\sigma_\delta^2 \frac{\partial^2 g_1}{\partial p_z^2} + p_z \frac{\partial g_1}{\partial p_z} + g_1\right]$$

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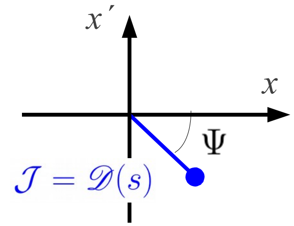

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Synchrotron motion

Contribution of dipolar wakefield

Fokker-Planck dynamics (damping and diffusion)

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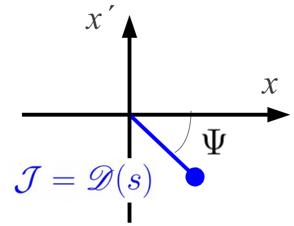

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Imaginary part of Ω determines stability of perturbation

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Matrix theory of transverse collective instabilities

The linear problem can be solved by expanding the perturbation in terms of orthogonal modes (Sacherer's method)

Gauss-Laguerre modes

Scaled action $\mathcal{I}/\langle\mathcal{I}\rangle$

$$g_1(\Phi, r) = \sum_{q,n} a_q^n g_q^n(r) \frac{e^{-r}}{2\pi} e^{in\Phi} = \sum_{q=0}^{\infty} \sum_{n=-q}^{\infty} a_q^n \boxed{\frac{r^{n/2} L_q^n(r)}{\sqrt{(q+n)!/q!}} \frac{e^{-r}}{2\pi}} e^{in\Phi}$$

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The linear problem can be solved by expanding the perturbation in terms of orthogonal modes (Sacherer's method)

Scaled action $\mathcal{I}/\langle\mathcal{I}\rangle$

Gauss-Laguerre modes

$$g_1(\Phi, r) = \sum_{q,n} a_q^n g_q^n(r) \frac{e^{-r}}{2\pi} e^{in\Phi} = \sum_{q=0}^{\infty} \sum_{n=-q}^{\infty} a_q^n \boxed{\frac{r^{n/2} L_q^n(r)}{\sqrt{(q+n)!/q!}} \frac{e^{-r}}{2\pi}} e^{in\Phi}$$

So that the linear problem reduces to the matrix equation

Coupling matrix associated with dipolar impedance

$$\left[\frac{\Omega - m\omega_s}{c} + \frac{i}{c\tau_x} + \frac{i(2p+m)}{c\tau_z}\right] a_p^m + \frac{2\pi I}{\gamma I_A} \int dk \sum_{n,q} \boxed{\mathsf{D}_{p,q}^{m,n}(k + k_\xi)} a_q^n = \frac{i}{2c\tau_z} \left(R_p^m a_{p+1}^{m-2} + T_p^m a_{p-1}^{m+2}\right)$$

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Mode coefficient of (azimuthal,radial) mode number (m,p), etc.

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Mode-dependent synchrotron damping

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This is an eigenvalue problem: truncating and numerically solving it gives normal modes that are linear superpositions of the a_p^m's, each with a complex frequency Ω.

If Ω has a positive imaginary part then the system is unstable

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Physical picture of Fokker-Planck dissipation

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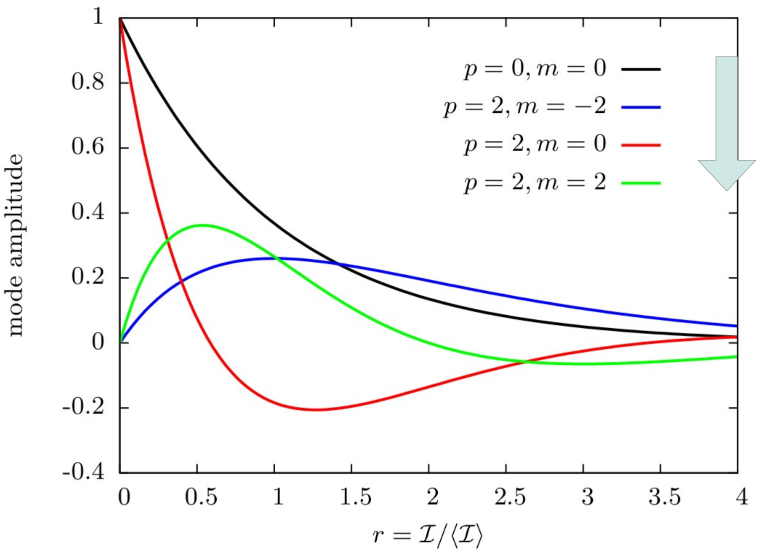

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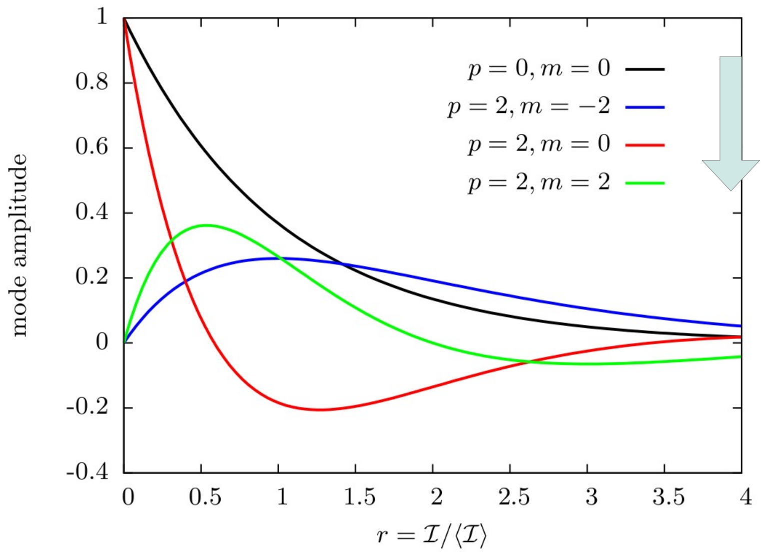
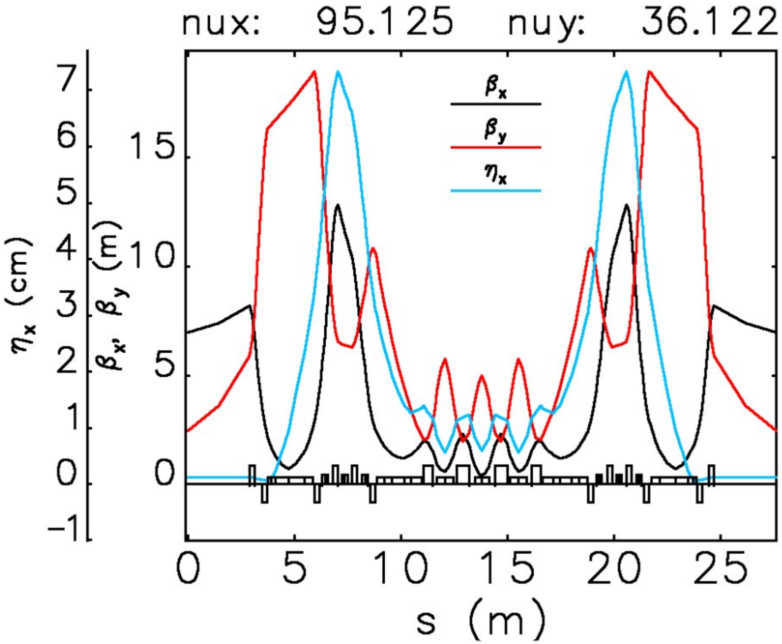


Increasing effective damping rate

- Diffusion also results in additional coupling between modes, but this is weak

Application to APS-U 7-bend achromat lattice with resistive wall transverse impedance model



Parameters
($V_{\mathrm{rf}} = 4.1$ MV)

$$\gamma = 6\ \mathrm{GeV}/mc^2$$
$$\mathcal{C}_R = 1104\ \mathrm{m}$$
$$\alpha_c = 5.66 \times 10^{-5}$$
$$\omega_s = 3271\ \mathrm{Hz}$$
$$\sigma_\delta = 0.0955\,\%$$
$$\tau_z = 14.06\ \mathrm{ms}$$
$$\varepsilon_0 = 67\ \mathrm{pm}$$
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M. Borland *et al*. Proc. IPAC15, 1776–1779 (2015); L. Farvacque et al. Proc. 2013 IPAC, 79 (2013).

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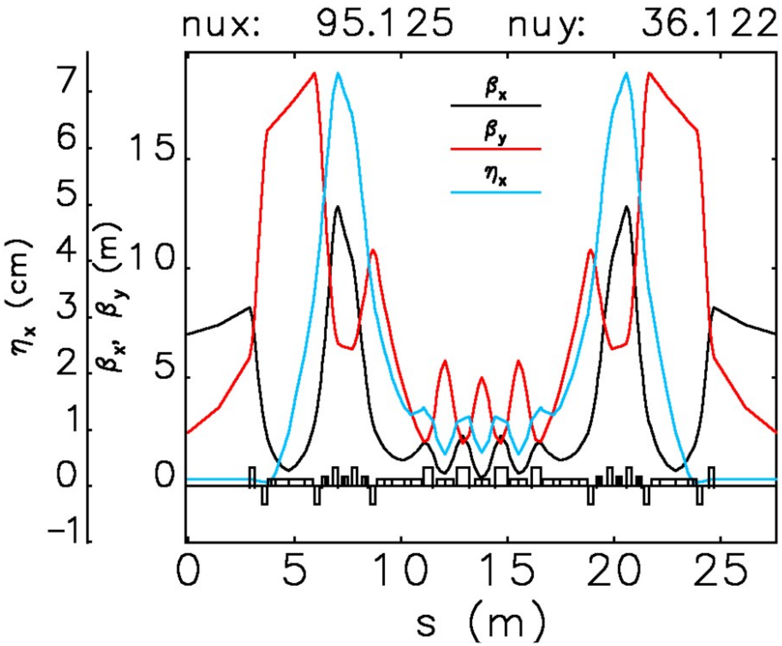

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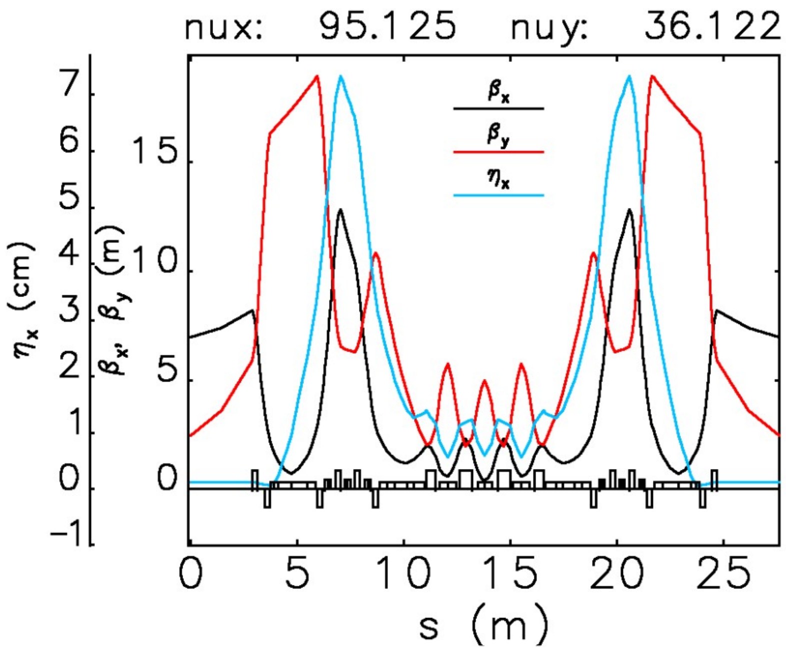

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Element	$\langle\beta_x\rangle$ (m)	b (mm)	ρ ($\Omega \cdot$ m)	L (m)
Flat IDs	7	3	3.16×10^{-8}	150
Round IDs	7	3	3.16×10^{-8}	25
ID transition	8	$11 \to 3$	3.16×10^{-8}	6.3
ID transition	8	$3 \to 11$	3.16×10^{-8}	6.3
Al chamber	10	11	3.16×10^{-8}	605
Cu chamber	3	11	1.68×10^{-8}	224
SS 314L chamber	9	11	95.2×10^{-8}	85

$$\beta_x Z_D(k) = \eta_D \oint ds\ \beta_x(s) \frac{\text{sgn}(k) - i}{\pi b(s)^3} \sqrt{\frac{Z_0 \rho(s)}{2\,|k|}}$$

$$\text{round} : \eta_D = 1$$

$$\text{flat} : \eta_D = \pi^2/24$$

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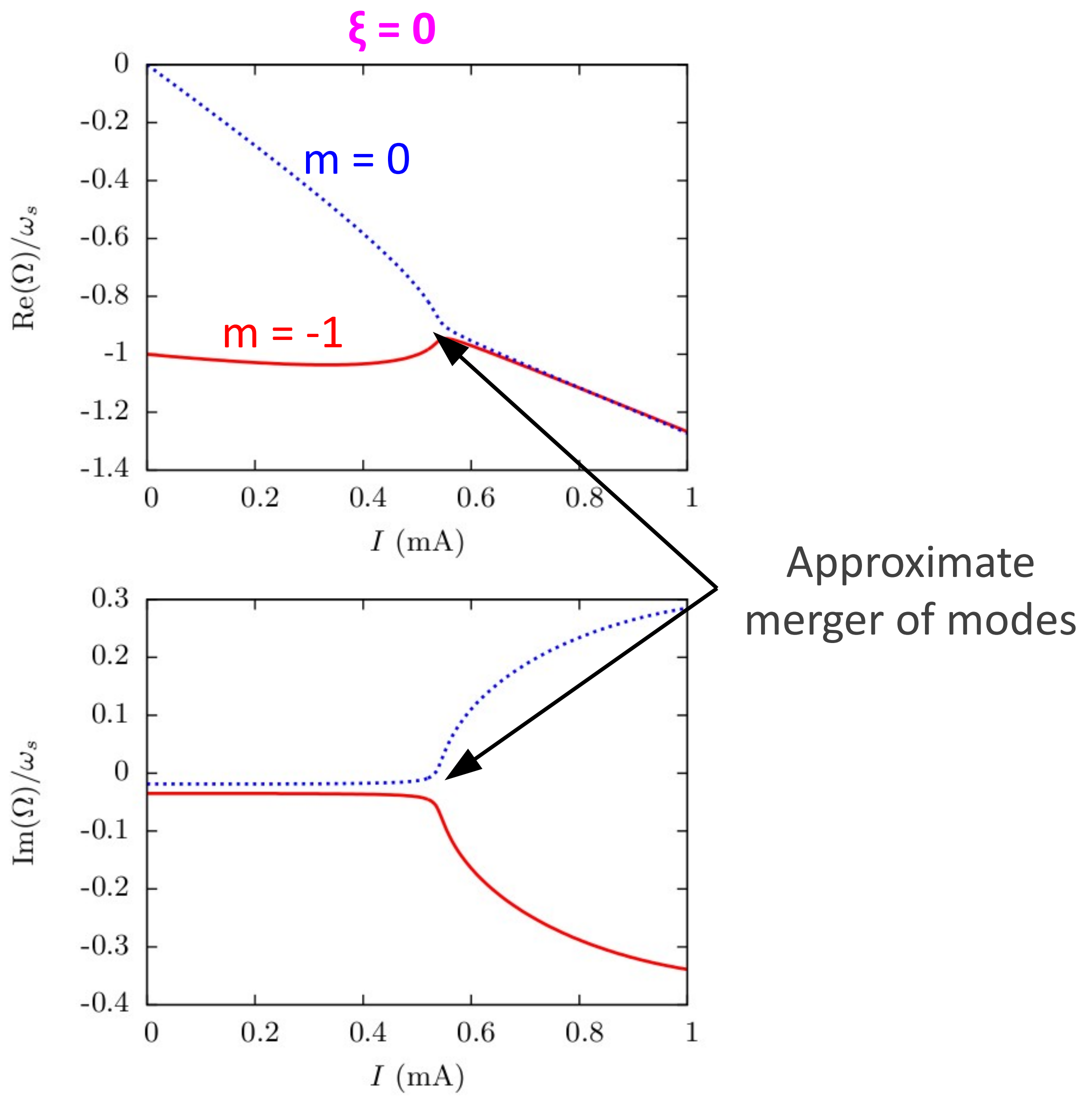
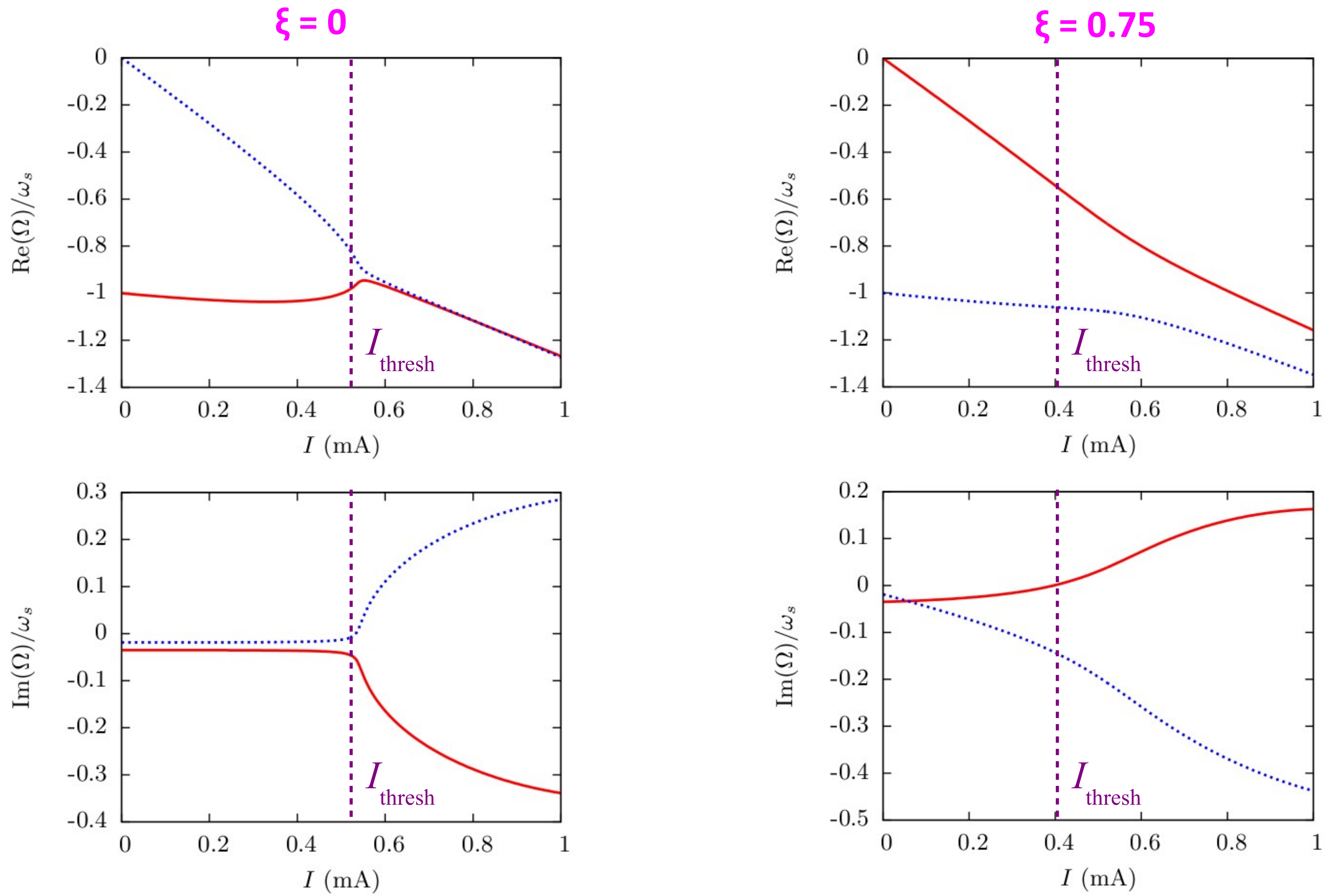
Mode coupling at zero chromaticity is very similar to that of Vlasov theory

In Vlasov picture the matrices are purely real at zero chromaticity, and two distinct real eigenvalues collide to become complex conjugates of each other



Mode coupling is less clear for non-zero chromaticity



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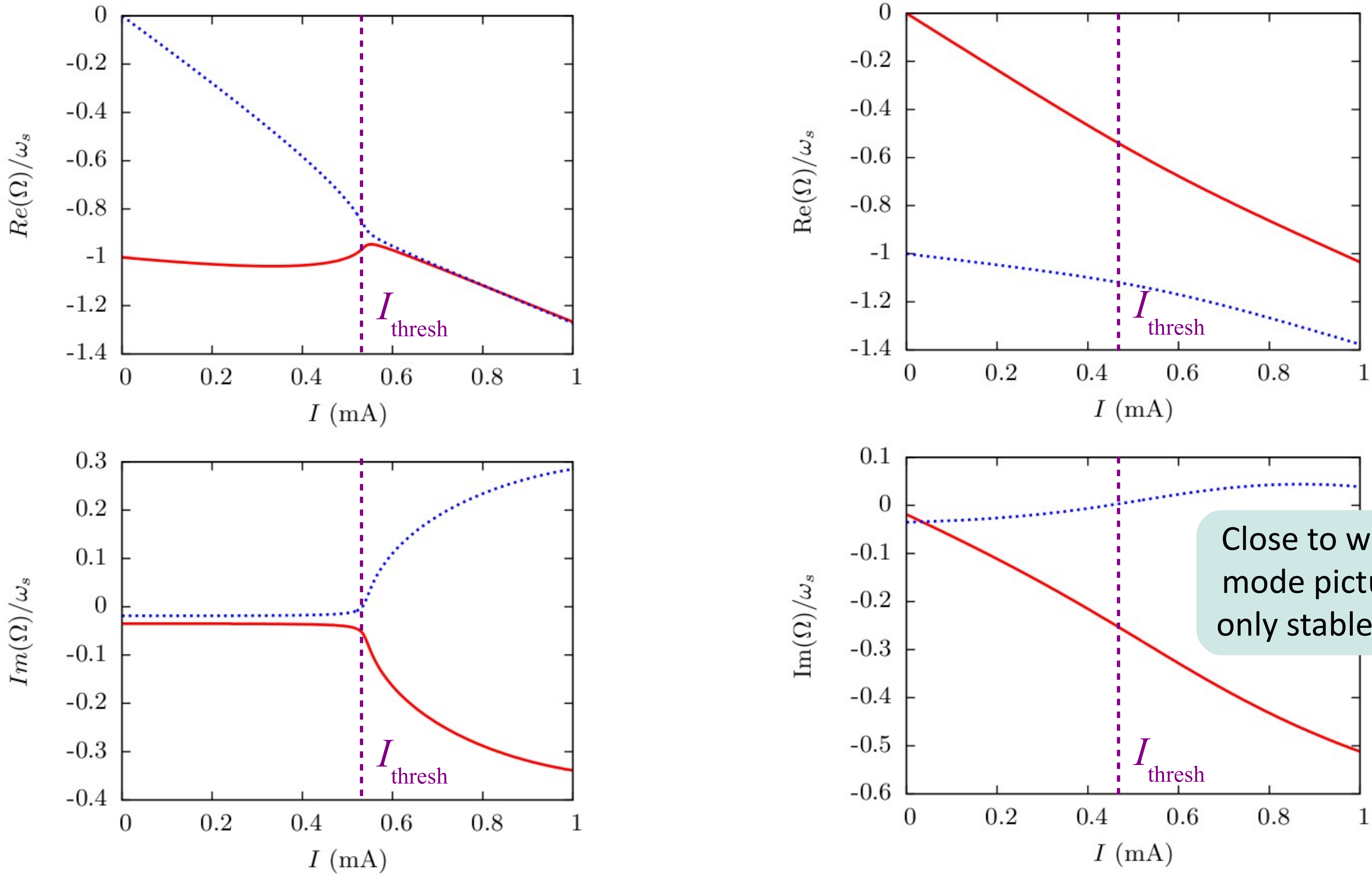
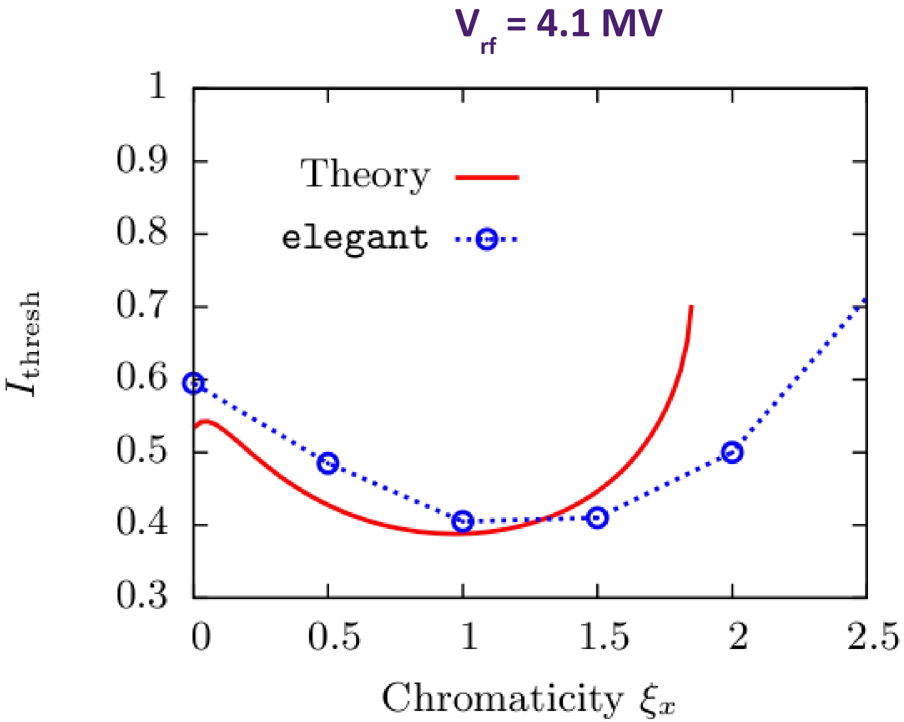
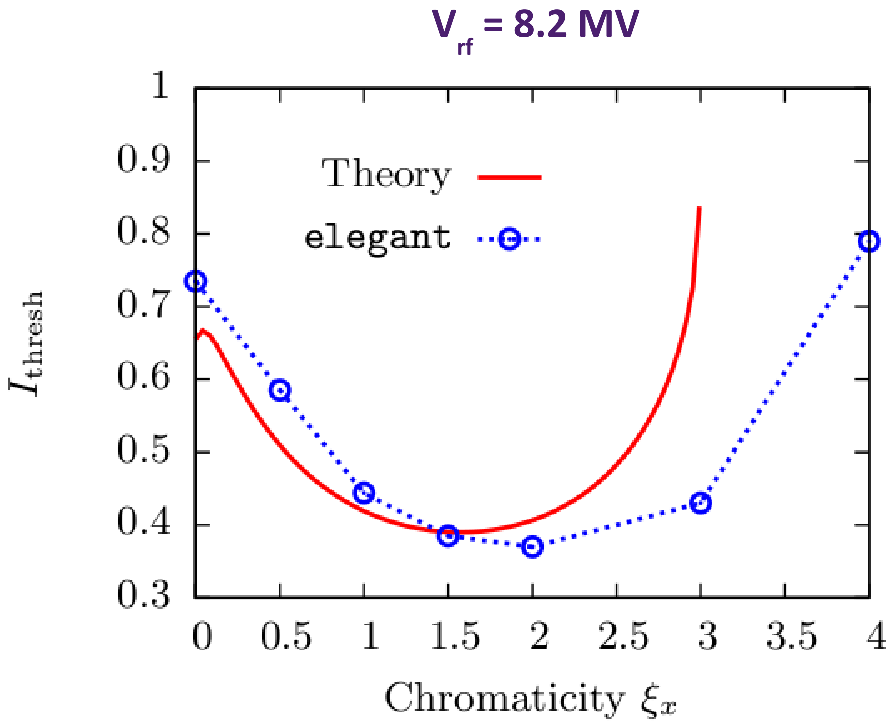
ξ = 0 ξ = 1.5



Close to where 2-mode picture has only stable modes

Usual coupled-mode theory of two synchrotron modes describes physics at low chromaticity





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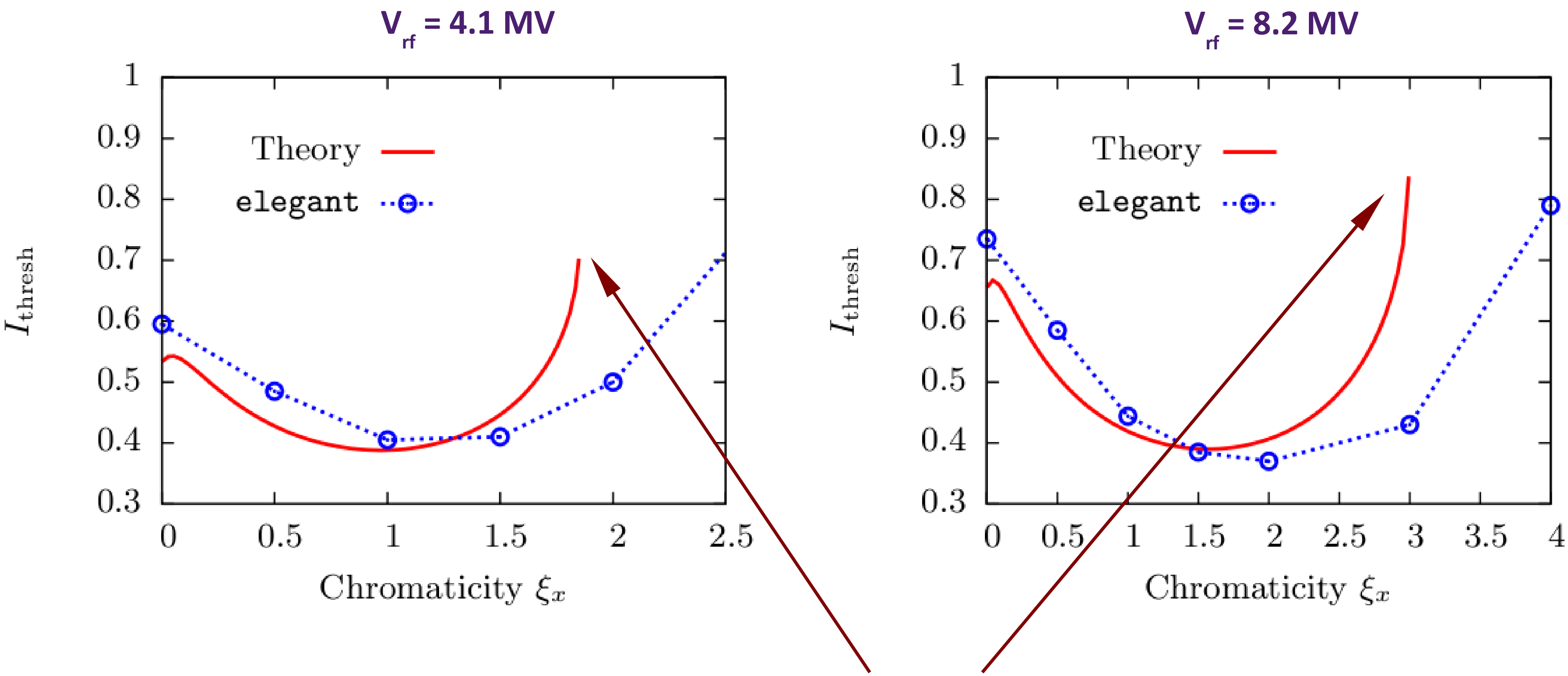
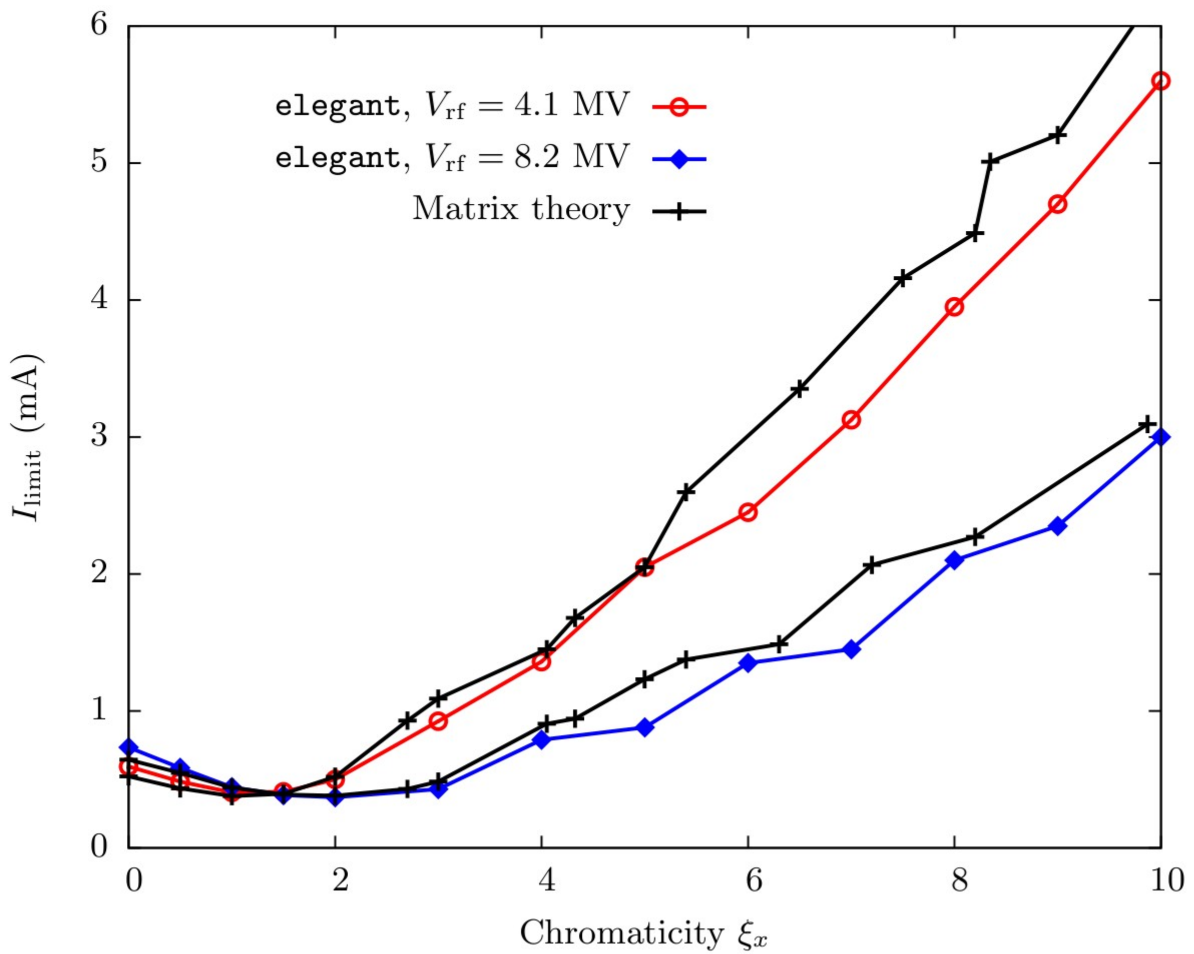


BUT, two-mode theory has no unstable root if

$$\xi_x \frac{\omega_0 \sigma_z}{\alpha_c c} \gtrsim 0.7$$

Instability threshold is well predicted by the Fokker-Planck theory by including many modes



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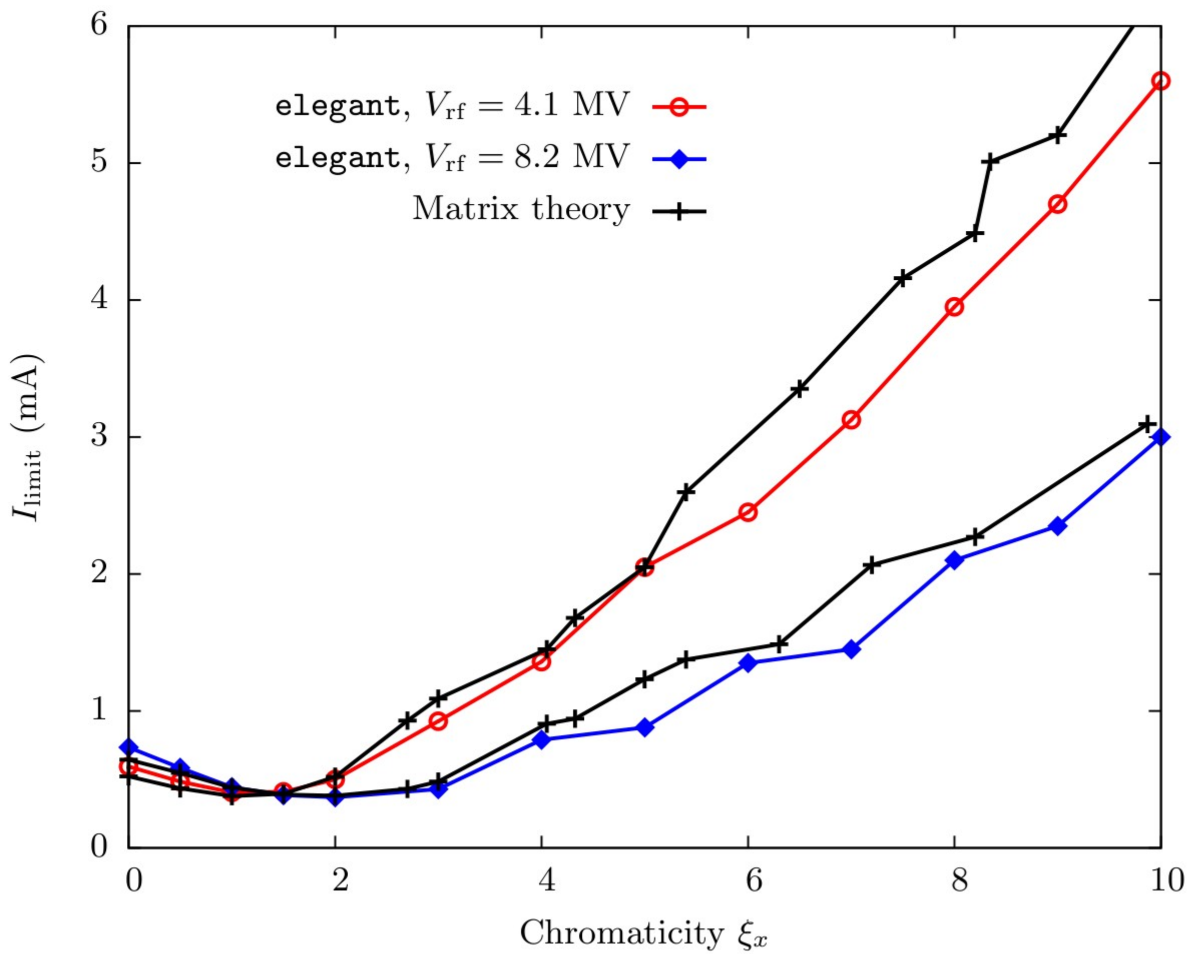

Landau damping is unimportant for these parameters

Artificially setting the nonlinear tune shift with amplitude to zero has essentially no effect in the simulation predictions

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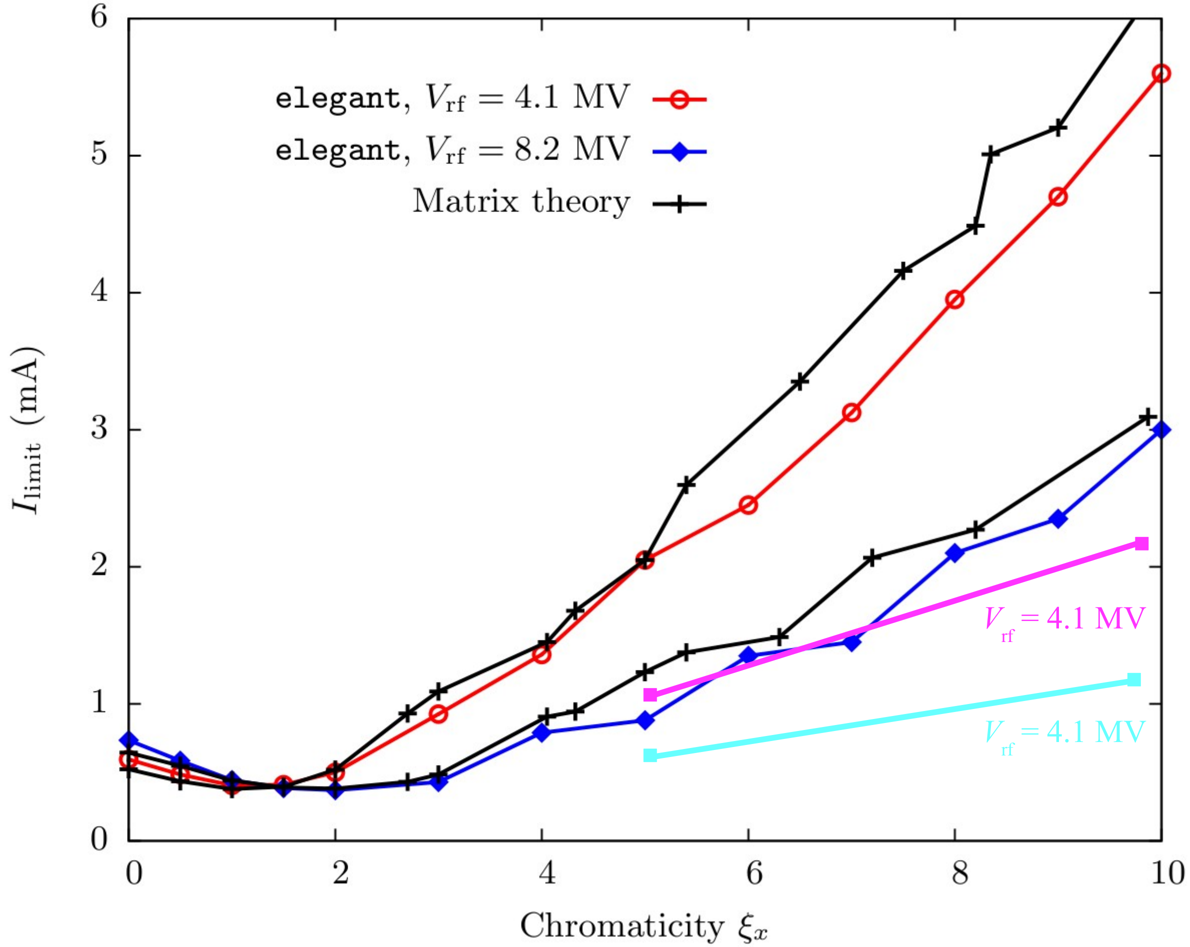

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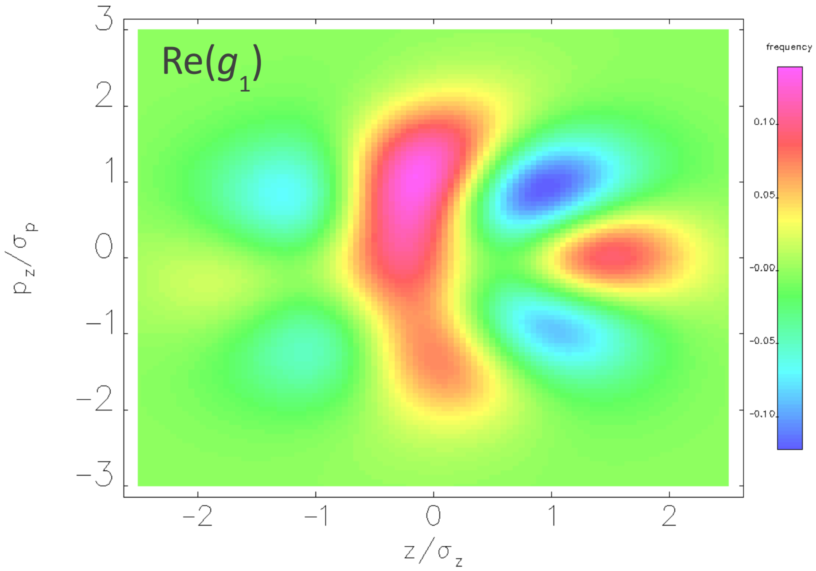
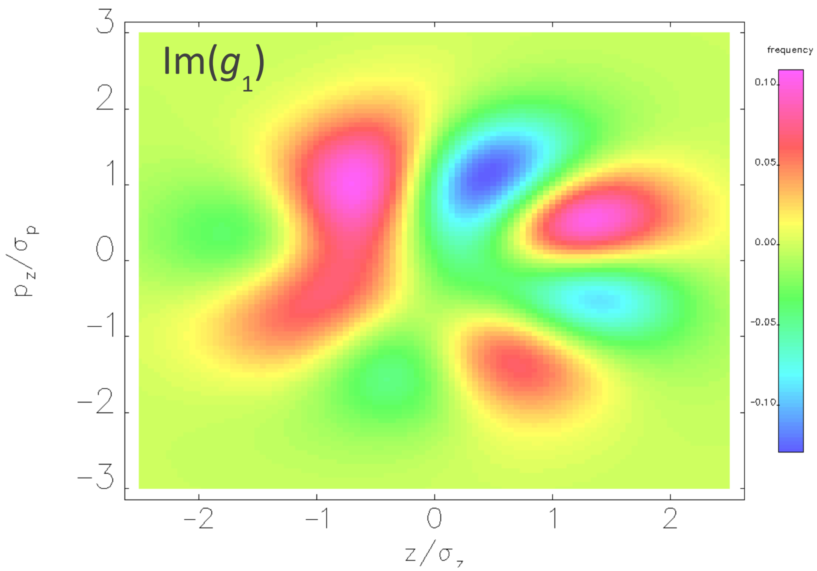
Vlasov theory underestimates I_{thresh} by a factor of 2 at high chromaticity.

Visualization of the unstable mode at ξ = 5

Theory





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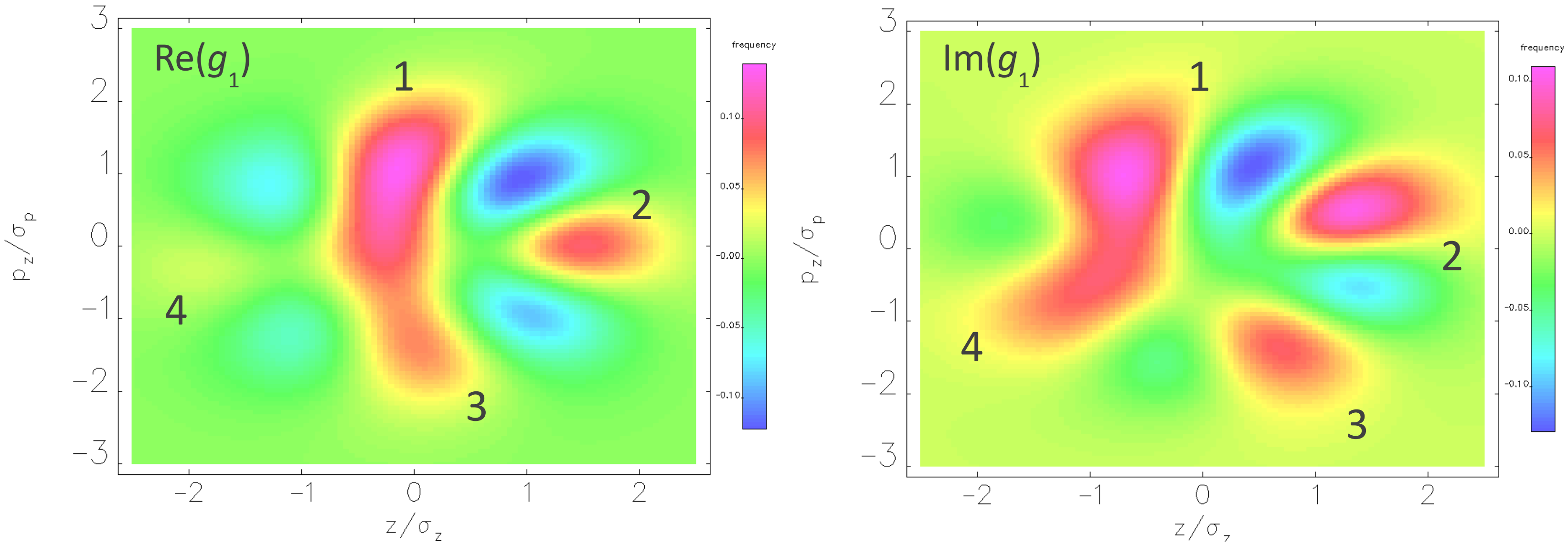
Comprised mostly of modes with m = -4



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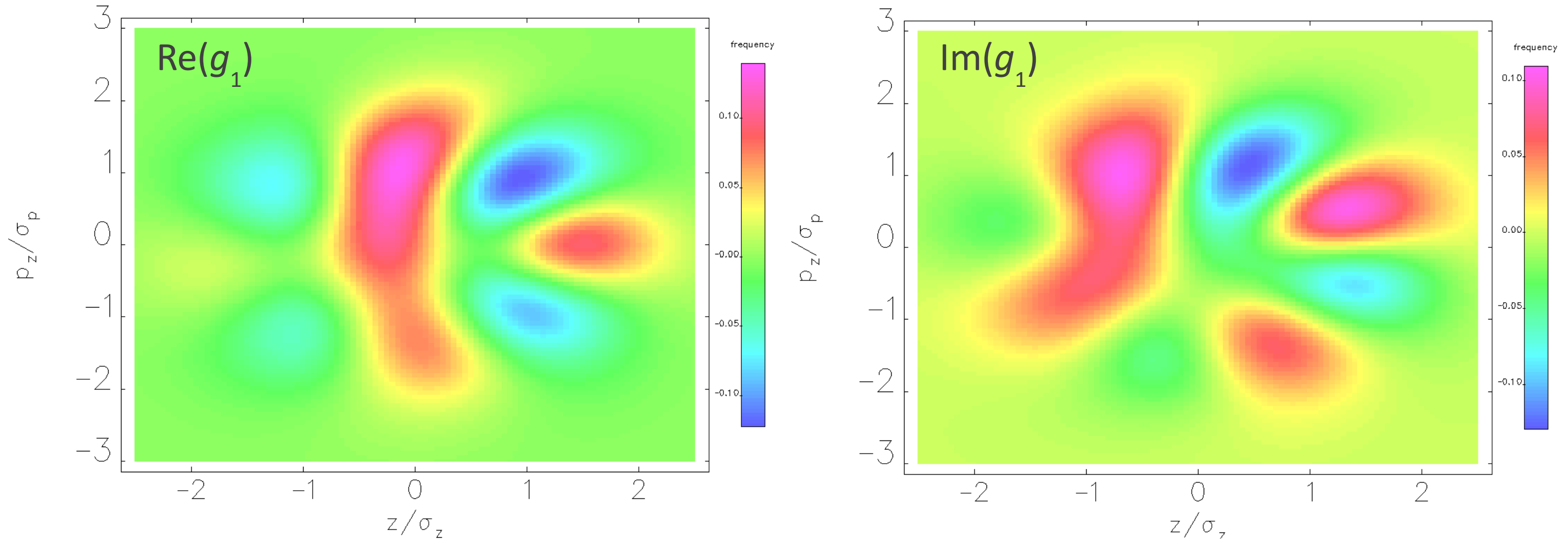

`elegant`
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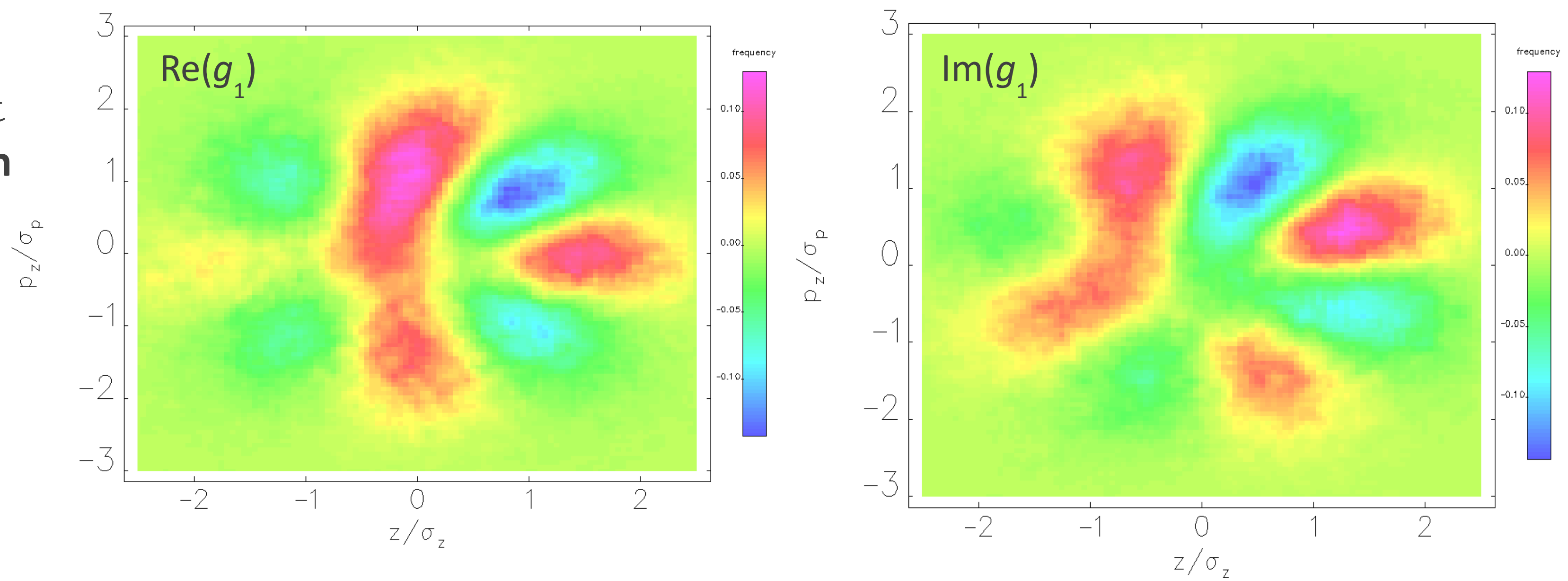

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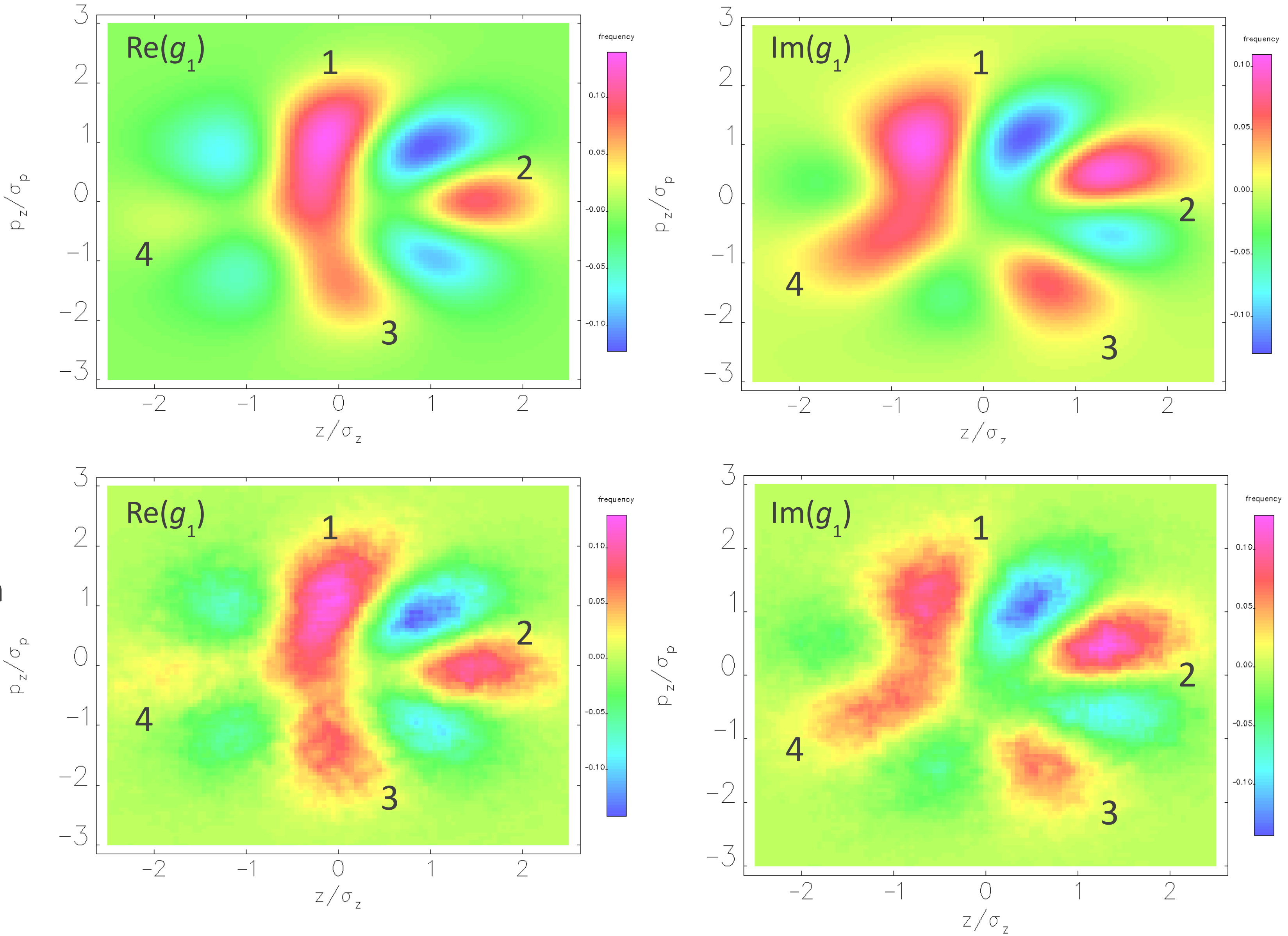
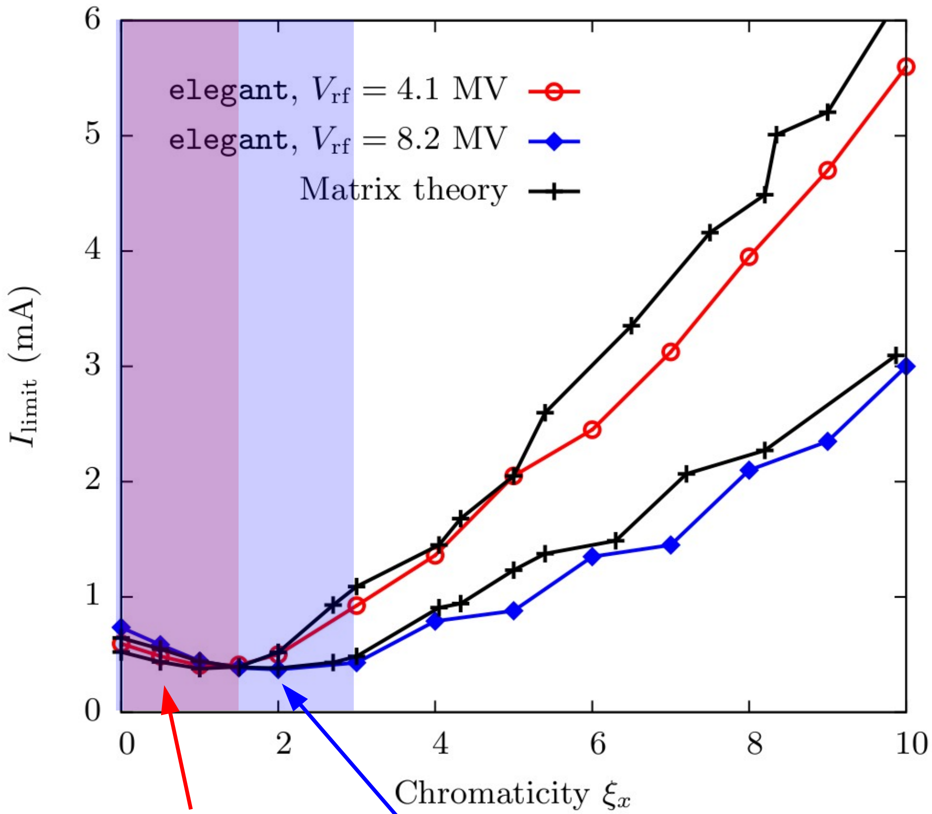


Physics of collective transverse instability



Region where 2-mode theory is valid for V_{rf} = 4.1 MV

Region where 2-mode theory is valid for V_{rf} = 8.2 MV

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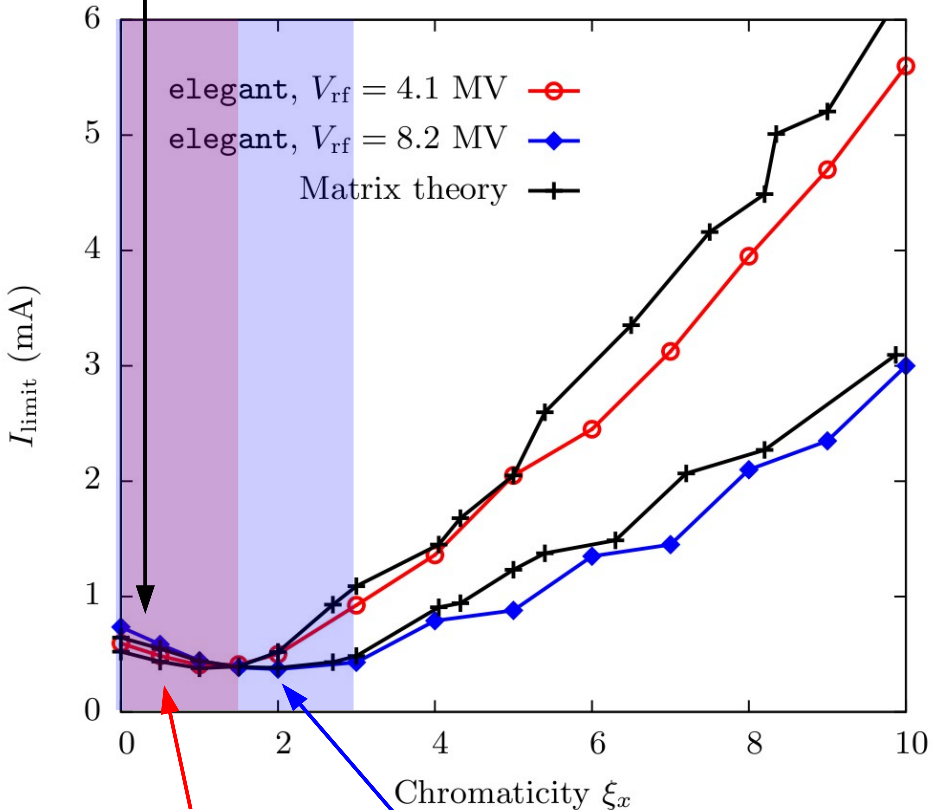

At very low chromaticity, higher rf voltage gives larger stable current:

Larger rf → Larger synchrotron frequency
→ Larger required frequency shift to merge modes
[Classic transverse mode coupling instability (TMCI)]

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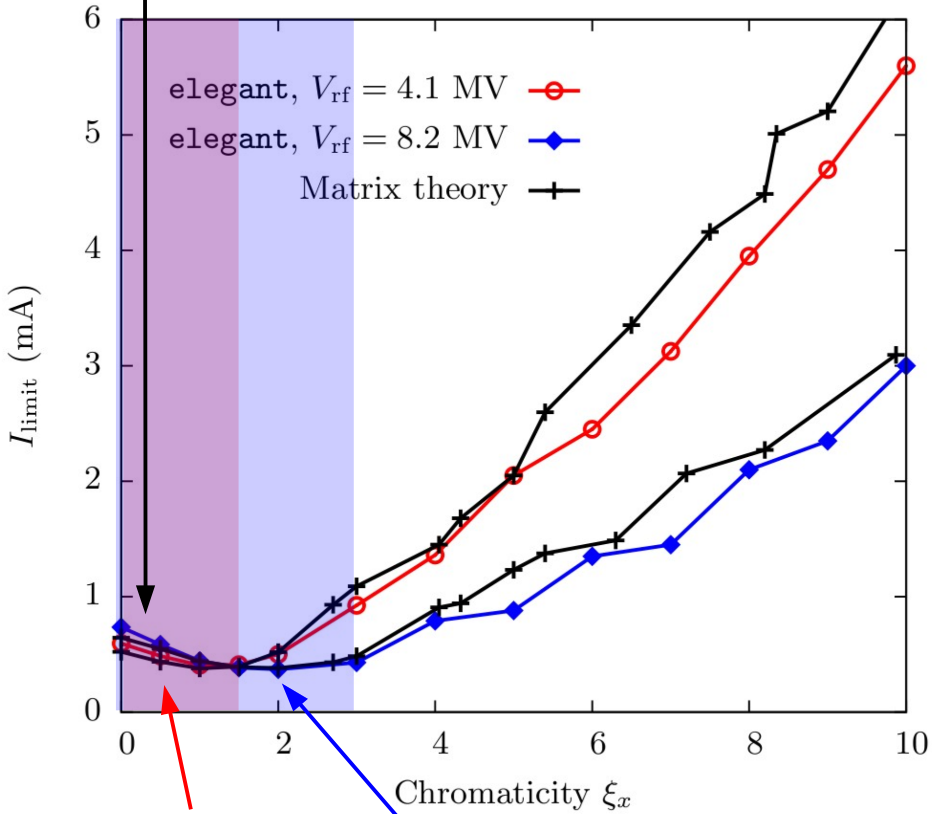

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Unstable eigenmode is comprised of many Gaussian-Laguerre basis modes, and higher-order modes have larger Fokker-Planck damping

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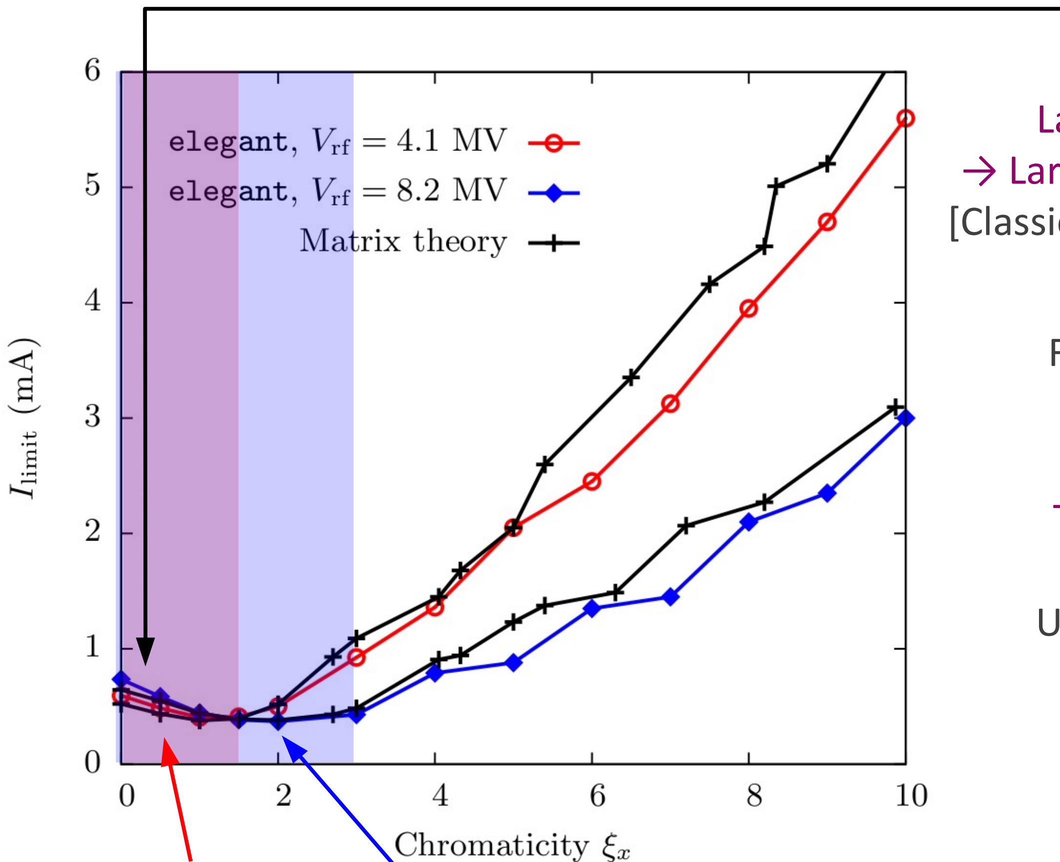

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We have also compared results for the "textbook" example of a constant wake function, finding qualitatively similar behavior

Conclusions & future work

- A Fokker-Planck analysis may be required to determine stability in storage rings with significant levels of synchrotron radiation when $\xi \neq 0$
- Damping and diffusion affects finer-scale perturbations more strongly, which results in larger effective damping rates for higher-order modes
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Thank you for your attention!