BI-COMPLEX TOOLBOX APPLIED TO GYROMAGNETIC BEAM BREAK-UP

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Overview

Transverse instability of a multi-bunch beam in the presence of a longitudinal magnetostatic field and hybrid dipole modes is considered analytically within a single-section model. It incorporates resonant interaction with beam harmonics and eigenmodes, degenerated waves of different polarizations, and the Lorentz' RF force contribution. The analysis is performed in a very compact form using a bi-complex *i,j*-space including four-component collective frequency of the instability. Rotating polarization of the collective field is determined by that is in agreement with experimental data. The other three components represent detuning of the collective frequency , the "left-hand", and "right-hand" increments of the gyro-magnetic BBU effect. The scalar hyper-complex toolbox can be applied to designing of non-ferrite non-reciprocal devices, spin transport, and for characterization of complex transverse dynamics in gyro-devices such as Gyro-TWTs.

Bi-complex transverse dynamics

$$\frac{1}{\gamma}\frac{d}{d\tau}\gamma\beta_z\frac{d}{d\tau}\tilde\rho + j\Omega\frac{d}{d\tau}\tilde\rho - \Omega_\perp\left|\tilde\rho\right|_j(1+j) = -i\frac{q_o}{2m_o\gamma}\sum_{r,\pm} D_{r,\pm}A_r\tilde C_r\exp\left(i(\theta_{r,\pm}\frac{\tau}{\tau_o}-\omega t)\right),$$

$$\tilde\rho = \rho_x + j\rho_y,\ \tilde C_r = C_r^{(x)} + jC_r^{(y)} \quad x = \mathrm{Re}_i\,\mathrm{Re}_j\,\tilde\rho \quad y = \mathrm{Re}_i\,\mathrm{Im}_j\,\tilde\rho$$

$$\tilde\rho(z,t) = \sum_{r,\pm}\frac{q_o\tau_o^{\ 2}}{2m_o\gamma}D_{r,\pm}\tilde\varphi_{r,\pm}(\tfrac{z}{L})\tilde C_r A_r\exp\left(i(h_r z - \tilde\omega t)\right) + \tilde\rho(t,0) + \tilde\rho'(t,0)\frac{1-\exp\left(-j\chi\Omega_o\right)}{j\Omega_o}$$

$$\tilde\varphi_{r,\pm}(\chi) = \frac{\exp(-i\theta_{r,\pm}\chi)}{\theta_{r,\pm}-ij\Omega_o}\left(\frac{\exp(i\theta_{r,\pm}\chi)-1}{i\theta_{r,\pm}} - \frac{1-\exp\left(-j\chi\Omega_o\right)}{j\Omega_o}\right)\quad \chi = z/L$$

Structure excitation

$$\frac{d\tilde C_r}{dt} - i(\omega-\omega_r)\tilde C_r = -\frac{1}{N_r}\int_V dV\ \vec j_\omega \tilde{\vec E}{}^0_{-r},$$

$$\tilde{\vec E}_r = \vec E_r^{\ (x)} + j\vec E_r^{\ (y)} \equiv \left(\vec E_r^{0(x)} + j\vec E_r^{0(y)}\right)\exp\left(ih_r z\right)$$

ODEs for modal Bi-Complex Amplitudes driving BBU

$$\left(\frac{d}{dt} + i\left(\omega_r - \omega\right)\right)\tilde C_r =$$

$$I\frac{A_r p_r}{4N_r}\frac{q_o\tau_o^{\ 2}L}{4m_o\gamma}\sum_{n=-\infty}^{+\infty}T_n\sum_{r',\pm}A_{r'}D_{r'\pm}\left[\Phi^{(1)\pm}_{rr'n}\tilde C_{r'}\exp(-in\omega_o t) + \Phi^{(2)\pm}_{rr'n}\tilde C^{*i}_{r'}\exp\left(i(2\omega'-n\omega_o)t\right)\right]$$

$$\tilde\Phi^{(1)\pm}_{rr'n} = \int_0^1 d\chi\,\tilde\varphi_{r,\pm}(\chi)\exp\left(i(nh_o \pm h_r - h_{r'})\chi L\right)\quad Si(x) = \sin(x)/x$$

$$\tilde\Phi^{(2)\pm}_{rr'n} = Si\left(T_o(2\omega'-n\omega_o)\right)\int_0^1 d\chi\,\tilde\varphi^{*i}_{r,\pm}(\chi)\exp\left(i(nh_o \mp h_r - h_{r'})\chi L\right)$$

Bi-Complex collective frequency of BBU $\tilde C_r(t) = \tilde C_{init}\exp\left(-i\omega''_r\tilde\nu t\right)$

$$\tilde\nu = i\left(\frac{I}{G_r}\tilde\Phi^{(1)+}_{rr0} - 1\right) + a_r$$

$$G_r = \omega''_r\left(\frac{A_r^{\ 2}p_r}{4N_r}\frac{q_o\tau_o^{\ 2}L}{4m_o\gamma}T_0 D_{r+}\right)^{-1} = \frac{m_o\gamma c^2}{q_o R_{\perp r}L/\lambda_r}\cdot\frac{4}{\pi}\cdot\frac{\beta_z^{\ 2}/\beta_r}{1-\beta_z\beta_r + \Xi_r(\beta_z-\beta_r)}$$

$$\tilde\Phi^{(1)\pm}_{rr0} = \frac{1}{\theta_{r,\pm}-ij\Omega_o}\left[\frac{1-\exp(-i\theta_{r,\pm})}{-i\theta_{r,\pm}}\left(\frac{1}{i\theta_{r,\pm}} + \frac{1}{j\Omega_o}\right) + \frac{1}{i\theta_{r,\pm}} + \frac{1-\exp\left(-i\theta_{r,\pm}-j\Omega_o\right)}{j\Omega_o(i\theta_{r,\pm}+j\Omega_o)}\right]$$

$$z = \tau\beta c \qquad \Omega_o = \Omega\tau_0$$

Bi-Complex slowly varying BBU amplitude

$$\tilde C_r = \tilde C_{init}\exp\left[-i\omega''_r\left(\mathrm{Re}_i\,\mathrm{Re}_j\,\tilde\nu + i\,\mathrm{Im}_i\,\mathrm{Re}_j\,\tilde\nu + j\,\mathrm{Re}_i\,\mathrm{Im}_j\,\tilde\nu + ij\,\mathrm{Im}_i\,\mathrm{Im}_j\,\tilde\nu\right)t\right] =$$

$$\tilde C_{init}\exp\left(i\left(\mathrm{Im}_i\,\mathrm{Re}_j\,\tilde\Phi^{(1)+}_{rr0}\frac{I}{G_r} - a_r\right)t_1\right)\cdot\exp\left(j\,\mathrm{Re}_i\,\mathrm{Im}_j\,\tilde\Phi^{(1)+}_{rr0}\frac{I}{G_r}t_1\right)\times$$

$$\left[\cosh\left(\mathrm{Im}_i\,\mathrm{Im}_j\,\tilde\Phi^{(1)+}_{rr0}\frac{I}{G_r}t_1\right) + ij\sinh\left(\mathrm{Im}_i\,\mathrm{Im}_j\,\tilde\Phi^{(1)+}_{rr0}\frac{I}{G_r}t_1\right)\right]\exp\left(\left(\mathrm{Re}_i\,\mathrm{Re}_j\,\tilde\Phi^{(1)+}_{rr0}\frac{I}{G_r} - 1\right)t_1\right)$$

Angular velocity of the collective field rotation

$$\mathrm{Im}_i\,\mathrm{Im}_j\,\tilde\omega = \mathrm{Re}_i\,\mathrm{Im}_j\,\tilde\Phi^{(1)+}_{rr0}\frac{I}{G_r}\omega''_r$$

Collective frequency detuning

$$\mathrm{Re}_i\,\mathrm{Re}_j\,\tilde\omega = \left(\mathrm{Im}_i\,\mathrm{Re}_j\,\tilde\Phi^{(1)+}_{rr0}\frac{I}{G_r} - a_r\right)\omega''_r$$

Maximum increment for two waves $\mathrm{Im}_i\,\mathrm{Re}_j\,\tilde\omega \pm \mathrm{Re}_i\,\mathrm{Im}_j\,\tilde\omega$

$$\mathrm{Im}_i\,\mathrm{Re}_j\,\tilde\omega + \left|\mathrm{Re}_i\,\mathrm{Im}_j\,\tilde\omega\right| = \left[\left(\mathrm{Re}_i\,\mathrm{Re}_j\,\tilde\Phi^{(1)+}_{rr0} + \left|\mathrm{Im}_i\,\mathrm{Im}_j\,\tilde\Phi^{(1)+}_{rr0}\right|\right)\frac{I}{G_r} - 1\right]\omega''_r$$

Metrics of commutative hypercomplex (scalar quaternion)

1	i	j	k
i	-1	k	$-j$
j	k	-1	$-i$
k	$-j$	$-i$	1

$$i^2 = -1,\ j^2 = -1,\ ij = ji \neq -1\ \ or\ \sqrt{-1}$$

$$\tilde a = \alpha_0 + i\alpha_1 + j\alpha_2 + ij\alpha_3$$

$$\left|\tilde a\right| \equiv N(\tilde a) = \sqrt[4]{\det\tilde a} \equiv \sqrt[4]{\tilde a\tilde a^{*i}\tilde a^{*j}\tilde a^{*i*j}} = \sqrt[4]{\left|\left|\tilde a\right|_j^2\right|_i^2}.$$

$$\det_i\tilde a = \left(\mathrm{Re}_i\,\tilde a\right)^2 + \left(\mathrm{Im}_i\,\tilde a\right)^2 = \tilde a\cdot\tilde a^{i*} \equiv \left|\tilde a\right|_i^2 = \alpha_0^2 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2 + 2j\left(\alpha_0\alpha_2 + \alpha_1\alpha_3\right)$$

Hyper zeros: $i \pm j,\ 1 \pm ij$

Inversed hypercomplex value $\tilde a^{-1} \equiv \frac{1}{\tilde a} = \frac{\tilde a^*}{\det\tilde a}$

$$\frac{1}{\tilde a} = \frac{\tilde a^{*j}}{\left|\tilde a\right|_j^2} \equiv \frac{\tilde a^{*i}}{\left|\tilde a\right|_i^2} \equiv \frac{\tilde a^{*i}}{\tilde a\cdot\tilde a^{*i}} = \frac{\tilde a^{*i}\cdot\left(\tilde a\tilde a^{*i}\right)^{*j}}{\tilde a\tilde a^{*i}\cdot\left(\tilde a\tilde a^{*i}\right)^{*j}} = \frac{\tilde a^{*i}\tilde a^{*j}\tilde a^{*i*j}}{\tilde a\tilde a^{*i}\tilde a^{*j}\tilde a^{*i*j}} = \frac{\tilde a^*}{\left|\tilde a\right|^4}.$$

Eulre's formula for hypercomplex:

$$\exp(ij\varphi) = \cosh\varphi + ij\sinh\varphi$$

$$\sqrt[n]{\tilde B} = \sqrt[n]{\left|\tilde B\right|}\exp\left(\frac{2\pi(ki+lj)}{n} + \frac{1}{n}ij\,\mathrm{arctanh}\left(\frac{c}{b}\right)\right)$$

Square root $\sqrt{(a+ijd)^2} = \begin{pmatrix}\pm a \pm ijd\\ \pm d \pm ija\end{pmatrix}$

$$\sqrt{\tilde B} = \pm\left|a^2-d^2\right|\cdot\left(\cosh\frac{\varphi}{2} + ij\sinh\frac{\varphi}{2}\right),\ where$$

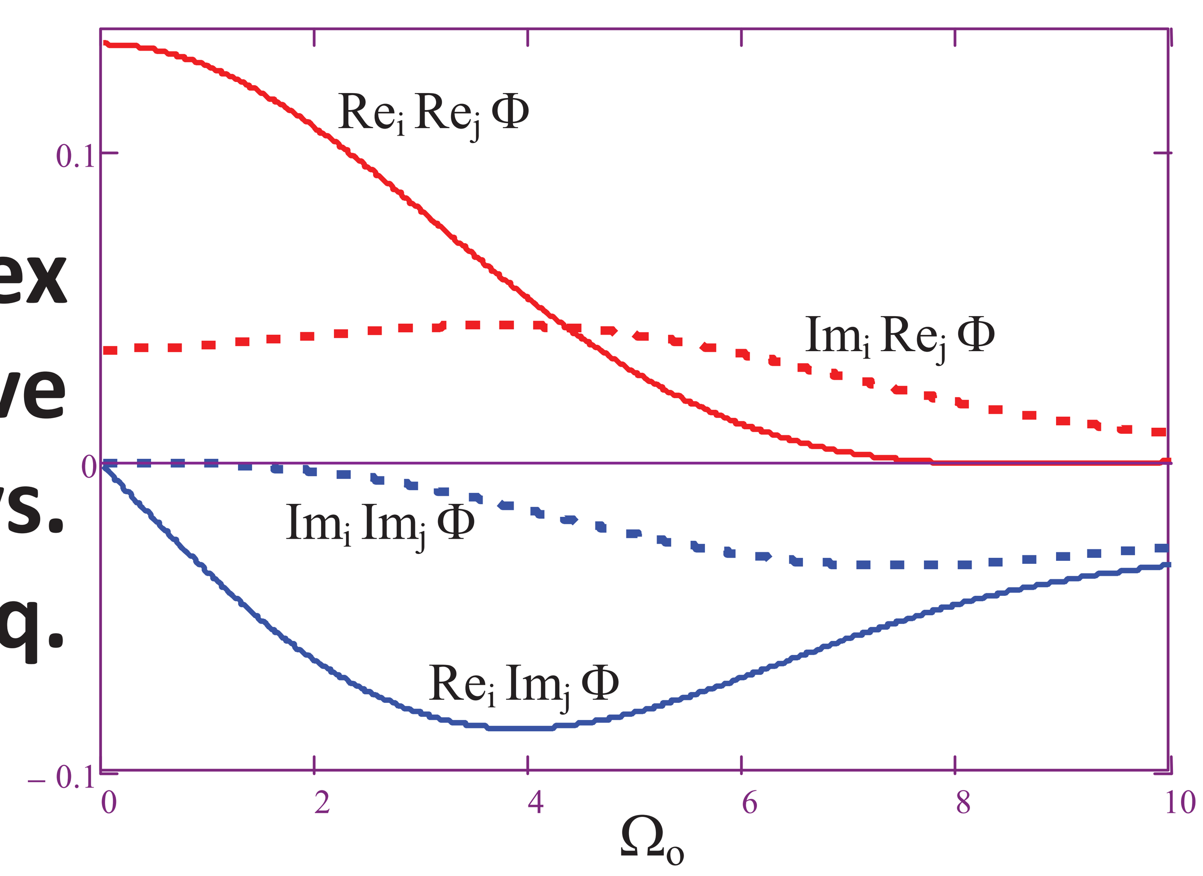
Bi-complex trigonometry

$$\tilde A = a + ijd = \left|a^2-d^2\right|\exp\left(ij\,\mathrm{arctanh}\frac{d}{a}\right)$$

$$\mathrm{arctanh}\,x \xrightarrow[|x|>1]{} -(i+j)\frac{\pi}{2} + \mathrm{arctanh}\frac{1}{x}$$

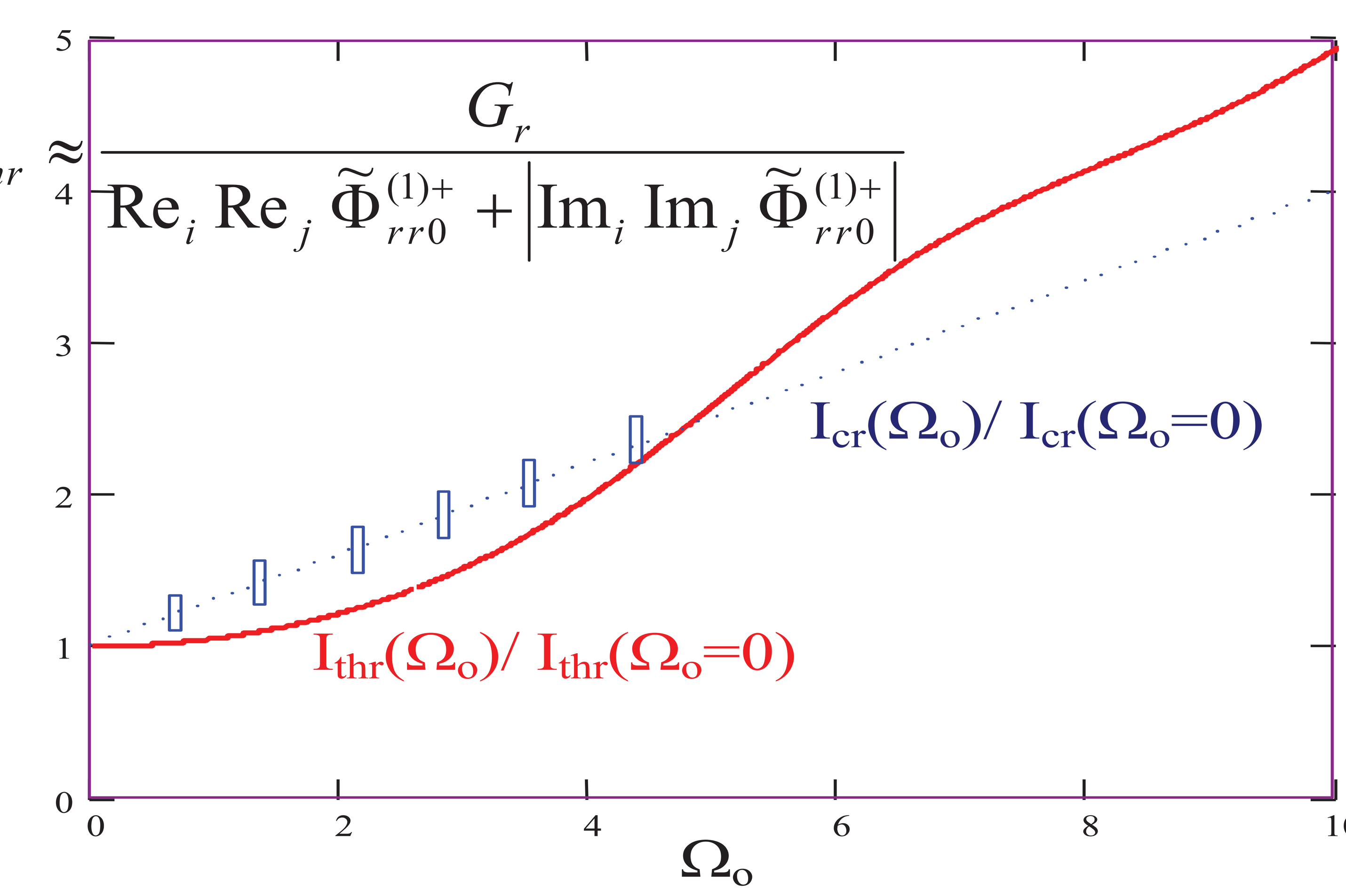
$$\varphi = \mathrm{arctanh}\left(\frac{2ad}{a^2+d^2}\right)$$

Bi-complex collective frequency vs. Larmor freq.



BBU threshold current

$$I_{thr} \approx \frac{G_r}{\mathrm{Re}_i\,\mathrm{Re}_j\,\tilde\Phi^{(1)+}_{rr0} + \left|\mathrm{Im}_i\,\mathrm{Im}_j\,\tilde\Phi^{(1)+}_{rr0}\right|}$$



Signal gain vs. detuning below threshold