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IMPLEMENTING THE FAST MULTIPOLE BOUNDARY ELEMENT METHOD WITH HIGH-ORDER ELEMENTS

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Abstract

The next generation of beam applications will require high-intensity beams with unprecedented control. For the new system designs, simulations that model collective effects must achieve greater accuracies and scales than conventional methods allow. The fast multipole method is a strong candidate for modeling collective effects due to its linear scaling. It is well known the boundary effects become important for such intense beams. We implemented a constant element fast boundary element method (FMBEM) [2] as our first step in studying the boundary effects. To reduce the number of elements and discretization error, our next step is to allow for curvilinear elements. In this paper, we will present our study on a quadratic and a cubic parametric method to model the surface.

Boundary Integral Equations

Equation 1: Discretized double layer potential for Dirichlet BC

$$\phi(\mathbf{x}_i) = \sum_{j=1}^{M} \int_{\Gamma_j} \frac{\partial G}{\partial n_y}(\mathbf{x}_i, \mathbf{y}_j)\eta(\mathbf{y}_j)\mathrm{d}\Gamma(\mathbf{y})$$

Equation 2: Discretized single layer field for Neumann BC

$$\frac{\partial \phi}{\partial n_x}(\mathbf{x}_i) = \sum_{j=1}^{M} \int_{\Gamma_j} \frac{\partial G}{\partial n_x}(\mathbf{x}_i, \mathbf{y}_j)\sigma(\mathbf{y}_j)\mathrm{d}\Gamma(\mathbf{y})$$

These formulations lead to well-conditioned systems but require accurate normals on all M elements. The curvilinear parametrization must reproduce the normal vectors to an acceptable tolerance.

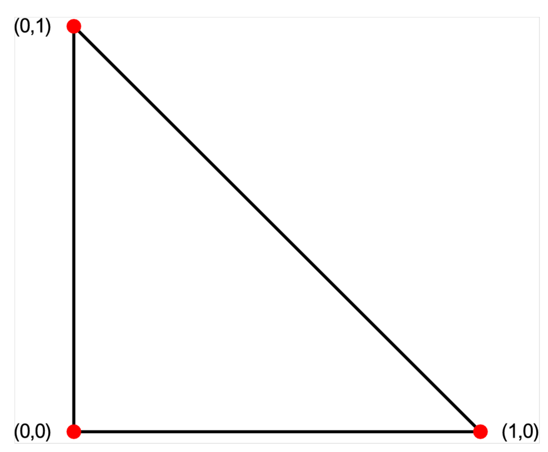

Figure 1: Parametric element in (u,v) with $\mathbf{x}_1 = \mathbf{x}(0,0)$, $\mathbf{x}_2 = \mathbf{x}(1,0)$, $\mathbf{x}_3 = \mathbf{x}(0,1)$ and corresponding normals.

$$(u,v)_{\text{quadratic}} \in [0,1] \times [0,1-u]$$

$$(u,v)_{\text{cubic}} \in [0,1] \times [0,1]: \quad (u,v)_{\text{cubic}} \to (r+s, \frac{s}{r+s}), (r+s) \neq 0$$

Vertices

$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$

Normals

$\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$

Equation 3: Quadratic parametric patch equations [3]

$$\mathbf{x}(u,v) = \mathbf{a}_{00} + \mathbf{a}_{10}u + \mathbf{a}_{01}v + \mathbf{a}_{20}u^2 + \mathbf{a}_{11}uv + \mathbf{a}_{02}v^2$$

$$\mathbf{d}_{ij} \equiv \mathbf{x}_j - \mathbf{x}_i$$

$$\mathbf{c}_{ij} \equiv \mathbf{c}(\mathbf{d}_{ij}, \mathbf{n}_i, \mathbf{n}_j)$$

$$\mathbf{a}_{00} = \mathbf{x}_1 \qquad \mathbf{a}_{20} = \mathbf{c}_{12}$$

$$\mathbf{a}_{10} = \mathbf{d}_{12} - \mathbf{c}_{12} \qquad \mathbf{a}_{11} = \mathbf{c}_{12} + \mathbf{c}_{13} - \mathbf{c}_{23}$$

$$\mathbf{a}_{01} = \mathbf{d}_{13} - \mathbf{c}_{13} \qquad \mathbf{a}_{02} = \mathbf{c}_{23}$$

$$\boldsymbol{\nu} = \frac{\mathbf{n}_i + \mathbf{n}_j}{2} \qquad d = \mathbf{d}_{ij} \cdot \boldsymbol{\nu} \qquad c = 1 - 2\Delta c$$

$$\Delta\boldsymbol{\nu} = \frac{\mathbf{n}_i - \mathbf{n}_j}{2} \qquad \Delta d = \mathbf{d}_{ij} \cdot \Delta\boldsymbol{\nu} \qquad \Delta c = \mathbf{n}_i \cdot \Delta\boldsymbol{\nu}$$

$$\mathbf{c}(\mathbf{d}_{ij}, \mathbf{n}_i, \mathbf{n}_j) = \begin{cases} \frac{\Delta d}{1-\Delta c}\boldsymbol{\nu} + \frac{d}{\Delta c}\Delta\boldsymbol{\nu} & c \neq \pm 1 \\ \mathbf{0} & c = \pm 1 \end{cases}$$

Equation 4: Cubic parametric patch equations [4]

$$\mathbf{x}(u,v) = \mathbf{a}_0(u) + \mathbf{a}_1(u)v + \mathbf{a}_2(u)v^2 + \mathbf{a}_3(u)v^3$$

$$\mathbf{x}_1 \to \mathbf{p}_{12}, \ \mathbf{x}_2 \to \mathbf{p}_{13}$$

$$\mathbf{n}_1 \to \mathbf{k}_{12}, \ \mathbf{n}_2 \to \mathbf{k}_{13}$$

$$\mathbf{p}(u) = \mathbf{a}_0 + \mathbf{a}_1 u + \mathbf{a}_2 u^2 + \mathbf{a}_3 u^3$$

$$\mathbf{k}(u) = \frac{1}{\left|\frac{d\mathbf{p}}{du}\right|^2}\left[\frac{d^2\mathbf{p}}{du^2} - \frac{\frac{d\mathbf{p}}{du}\cdot\frac{d^2\mathbf{p}}{du^2}}{\left|\frac{d\mathbf{p}}{du}\right|^2}\frac{d\mathbf{p}}{du}\right]$$

$$\alpha = \frac{(\mathbf{x}_2 - \mathbf{x}_1)\cdot\left[(\mathbf{n}_2\cdot\mathbf{n}_2)\mathbf{n}_1 + \frac{1}{2}(\mathbf{n}_1\cdot\mathbf{n}_2)\mathbf{n}_2\right]}{\frac{2}{3}(\mathbf{n}_1\cdot\mathbf{n}_1)(\mathbf{n}_2\cdot\mathbf{n}_2) - \frac{1}{6}(\mathbf{n}_1\cdot\mathbf{n}_2)}$$

$$\beta = \frac{-(\mathbf{x}_2 - \mathbf{x}_1)\cdot\mathbf{n}_2 - \frac{1}{3}(\mathbf{n}_1\cdot\mathbf{n}_2)\alpha}{\frac{2}{3}(\mathbf{n}_2\cdot\mathbf{n}_2)}$$

$$\mathbf{a}_0 = \mathbf{x}_1 \qquad \mathbf{a}_2 = \alpha\mathbf{n}_1$$

$$\mathbf{a}_1 = \mathbf{x}_2 - \mathbf{a}_0 - \mathbf{a}_2 - \mathbf{a}_3 \qquad \mathbf{a}_3 = \frac{1}{3}(\beta\mathbf{n}_2 - \mathbf{a}_2)$$

Table 1: Spherical Triangle Parameters

Vertices	(8.93, 4.49, 0.27) (3.08, 6.60, 6.85) (8.63, 0.54, 5.02)
Center	(7.75, 4.37, 4.56)
Radius	10
Area	28.970

Table 2: Percent Errors on Large Spherical Triangle

	Centroid	Unit Normal	Area
Quadratic	0.79	0.44	-0.33
Cubic	1.63	6.49	1.53
Truncated cubic	4.19	30.5	13.6

3rd order Taylor expansion of constructed cubic polynomial introduces error in the area and loses its interpolating property. The unit normal error is inconclusive due to surface distortion.

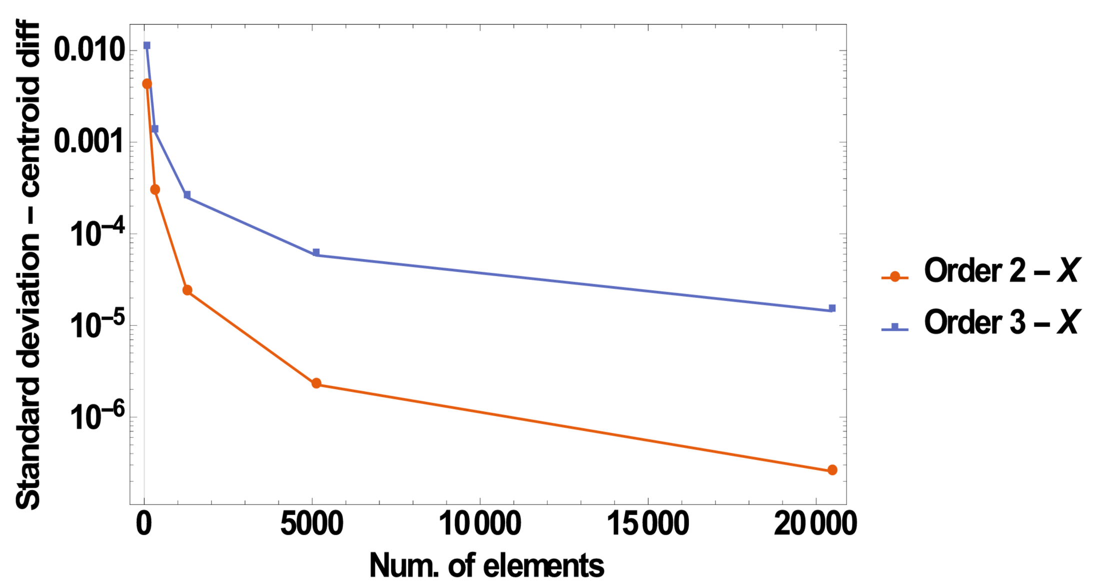

Figure 3a: Standard deviation of difference in centroid (X) vs. no. of elements for discretized sphere, comparing quadratic to truncated cubic patch.

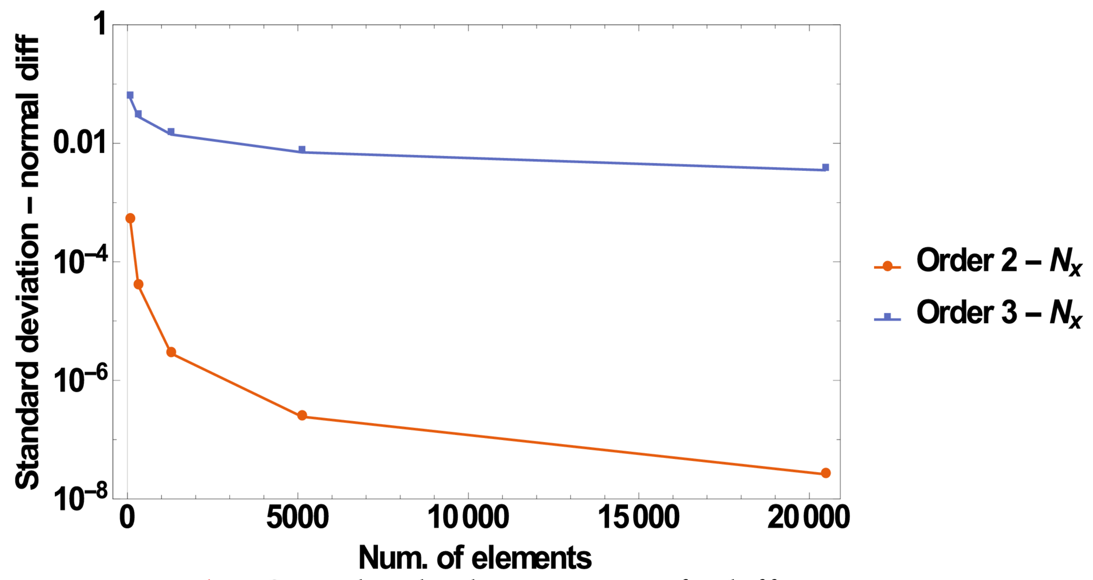

Figure 3b: Standard deviation of difference in unit normal (X) vs. no. of elements for discretized sphere. Error for truncated cubic patch may be due to distortion.

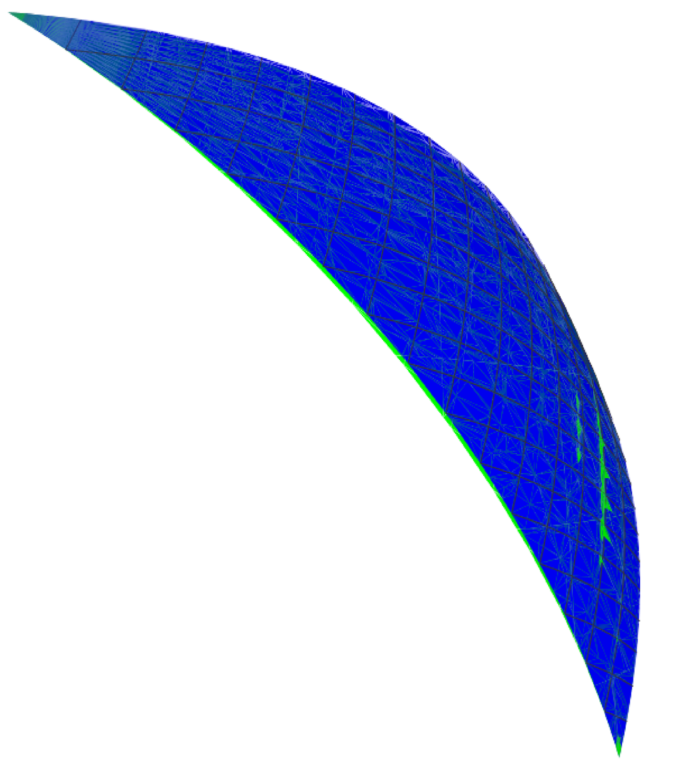

Figure 2a: Comparison of quadratic patch (blue) on spherical triangle (green).

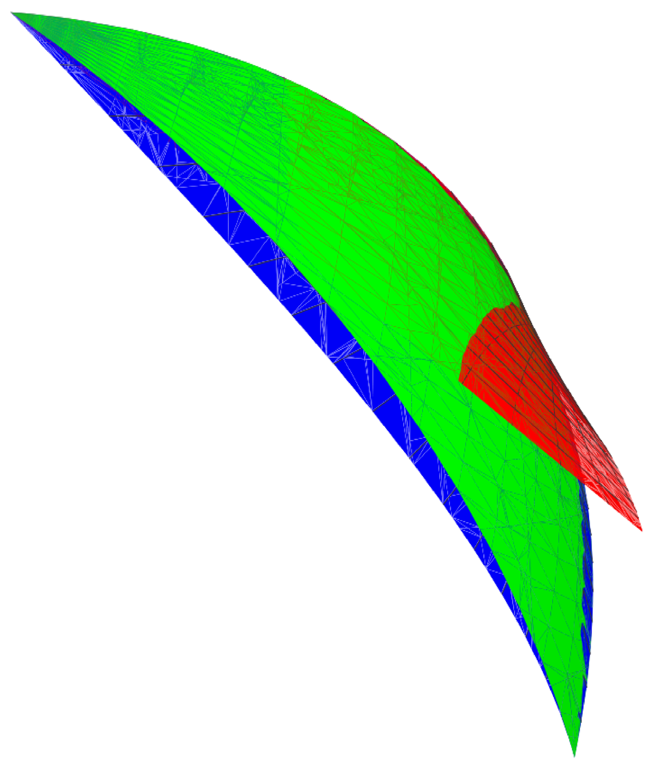

Figure 2b: Comparison of constructed cubic patch (blue) and its 3rd order Taylor expansion (red) on spherical triangle (green).

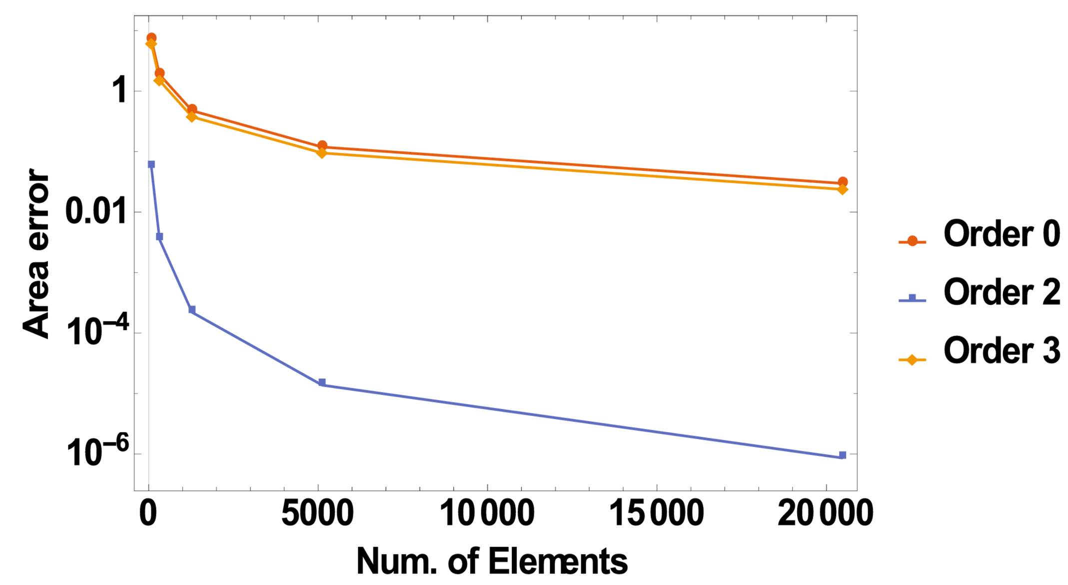

Figure 4: Percent error in area vs. no. of elements of discretized sphere. Quadratic patch shows the best quality.

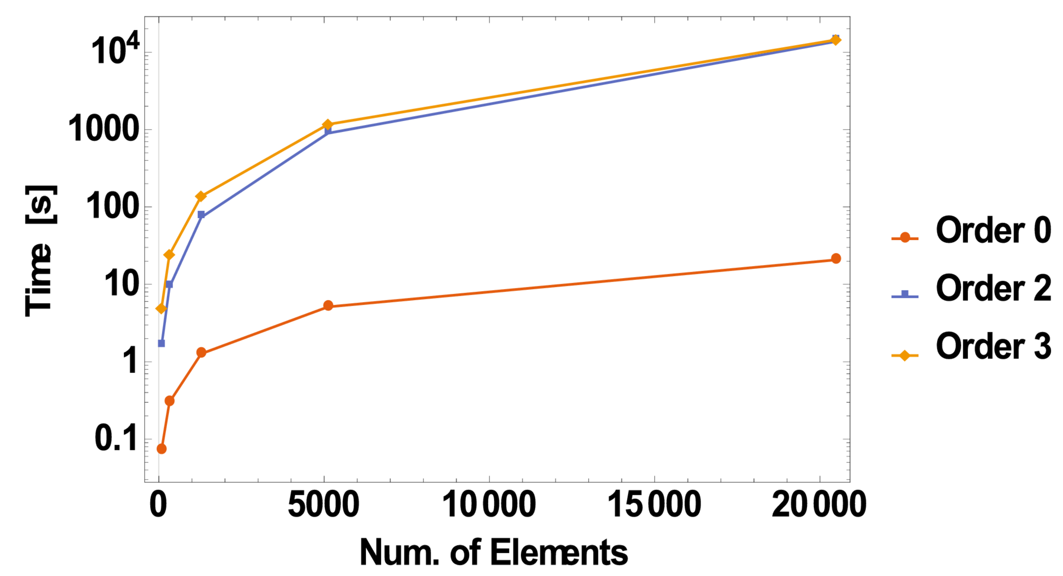

Figure 5a: Overall runtime vs. no. of elements. Runtime is dominated by integration.

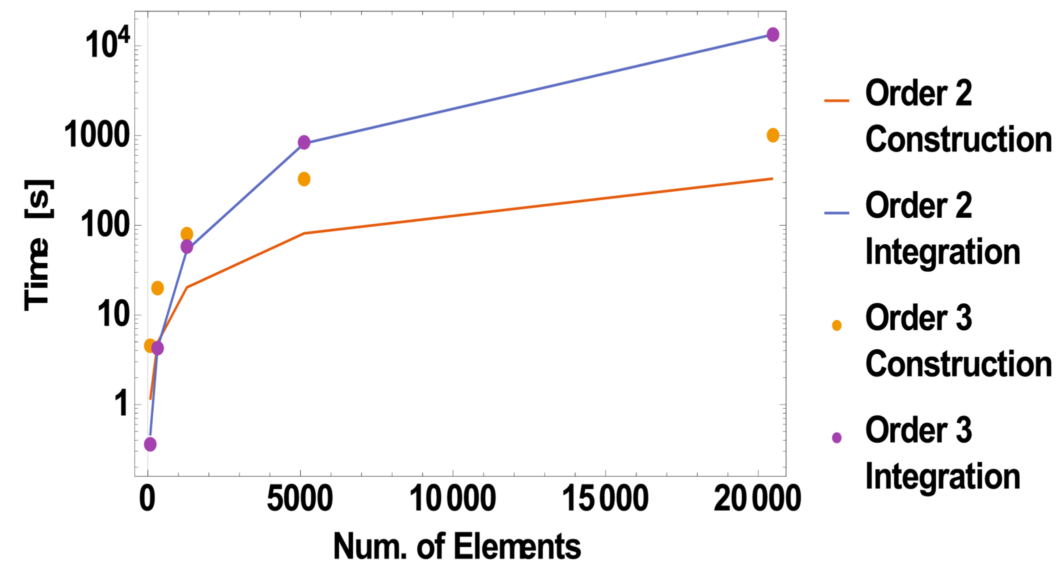

Figure 5b: Breakdown of runtime for quadratic and truncated cubic parametrization. Significant gains can be made by speeding up the integration.

Conclusion

We evaluated the quadratic parametric patch in [3] and the cubic patch in [4] as options for higher order discretization. Because we chose the well-conditioned double layer potential and single layer field integral equations, it is essential to accurately describe the normals across the structure's surface. We showed the quadratic patch gives adequate accuracy for large elements or small M. However, the truncated cubic patch needs to be studied further. We lose an essential interpolation property in the higher order terms, greatly reducing the patch quality. The runtime is mainly dominated by the integration, which we may improve using differential algebraic methods. Our next steps will be to allow for discontinuities such as edges or corners and decide whether element order >3 will be necessary for our purposes.

Acknowledgments

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