Sector Magnets or Transverse EM Fields in Cylindrical Coordinates [THPOA31]

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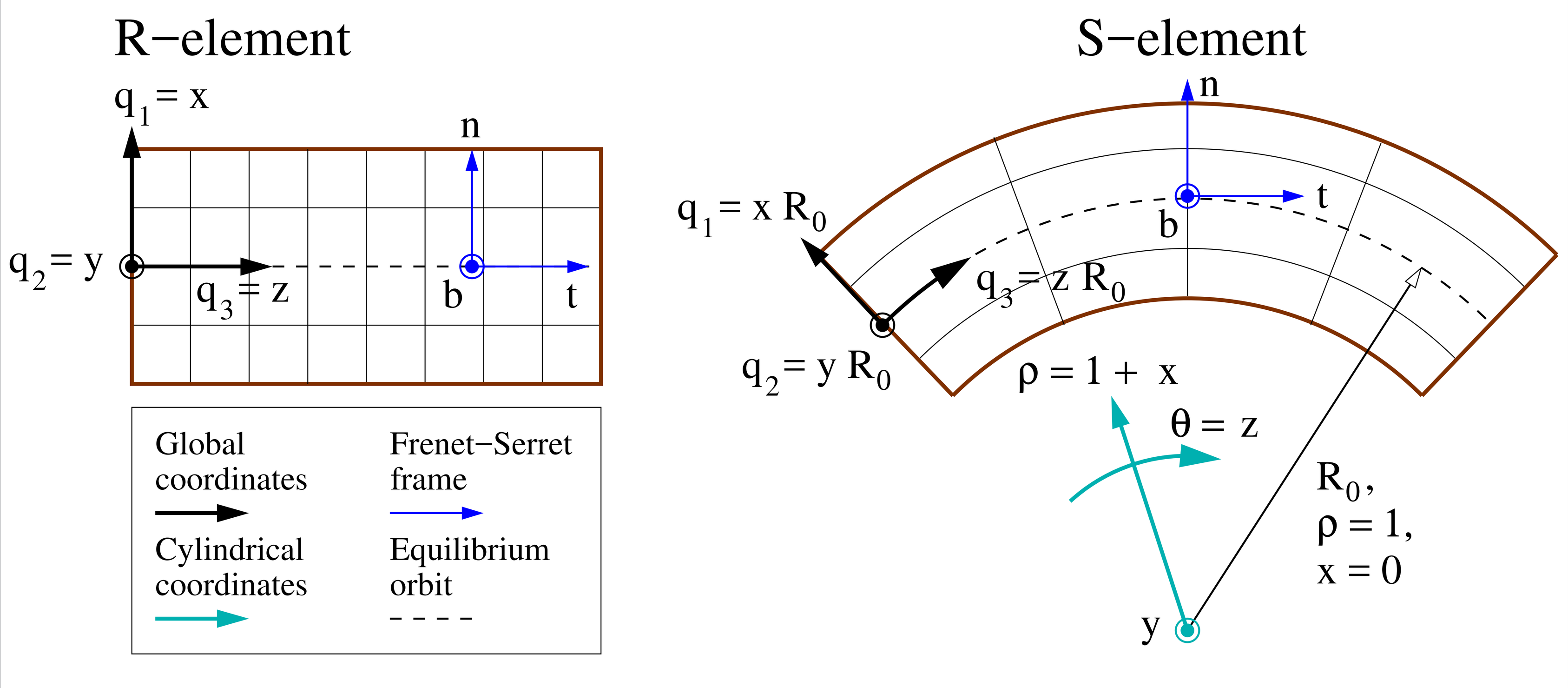
1. R- and S- elements



2. Multipoles in Cartesian Coordinates — Homogeneous Harmonic Polynomials

$$\triangle_\perp \Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0$$

$$\triangle_\perp \mathbf{A} = \left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2}\right)\hat{\mathbf{e}}_z = 0$$

Solutions are homogeneous harmonic polynomials of two variables

$$\mathcal{A}_n(x,y) = \Re\,\mathcal{Z}^n = \frac{1}{2}\left[(x+i\,y)^n + (x-i\,y)^n\right] = \sum_{k=0}^{n}\binom{n}{k} x^{n-k}y^k \cos\frac{k\,\pi}{2}$$

$$\mathcal{B}_n(x,y) = \Im\,\mathcal{Z}^n = \frac{1}{2\,i}\left[(x+i\,y)^n - (x-i\,y)^n\right] = \sum_{k=0}^{n}\binom{n}{k} x^{n-k}y^k \sin\frac{k\,\pi}{2}$$

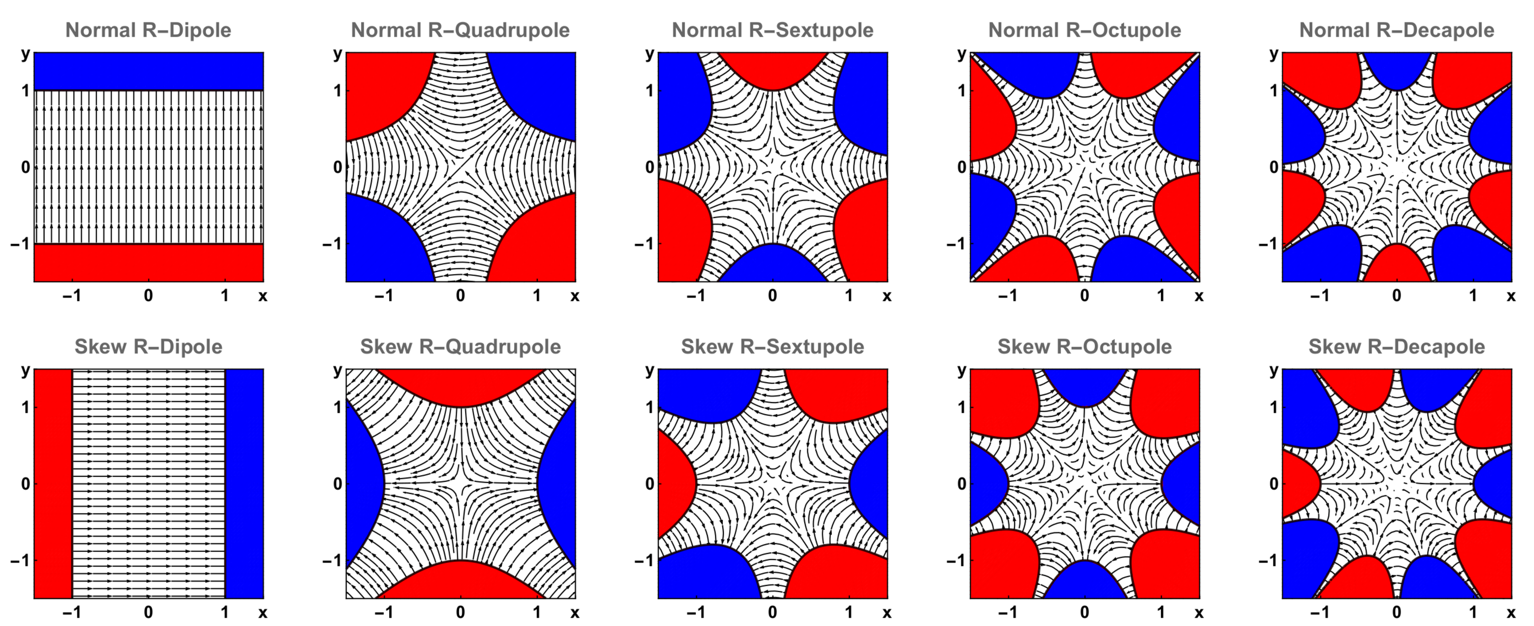
related to each other through the Cauchy-Riemann equation

$$\frac{\partial\,\mathcal{A}_n}{\partial x} = \frac{\partial\,\mathcal{B}_n}{\partial y} \qquad \frac{\partial\,\mathcal{A}_n}{\partial y} = -\frac{\partial\,\mathcal{B}_n}{\partial x}$$

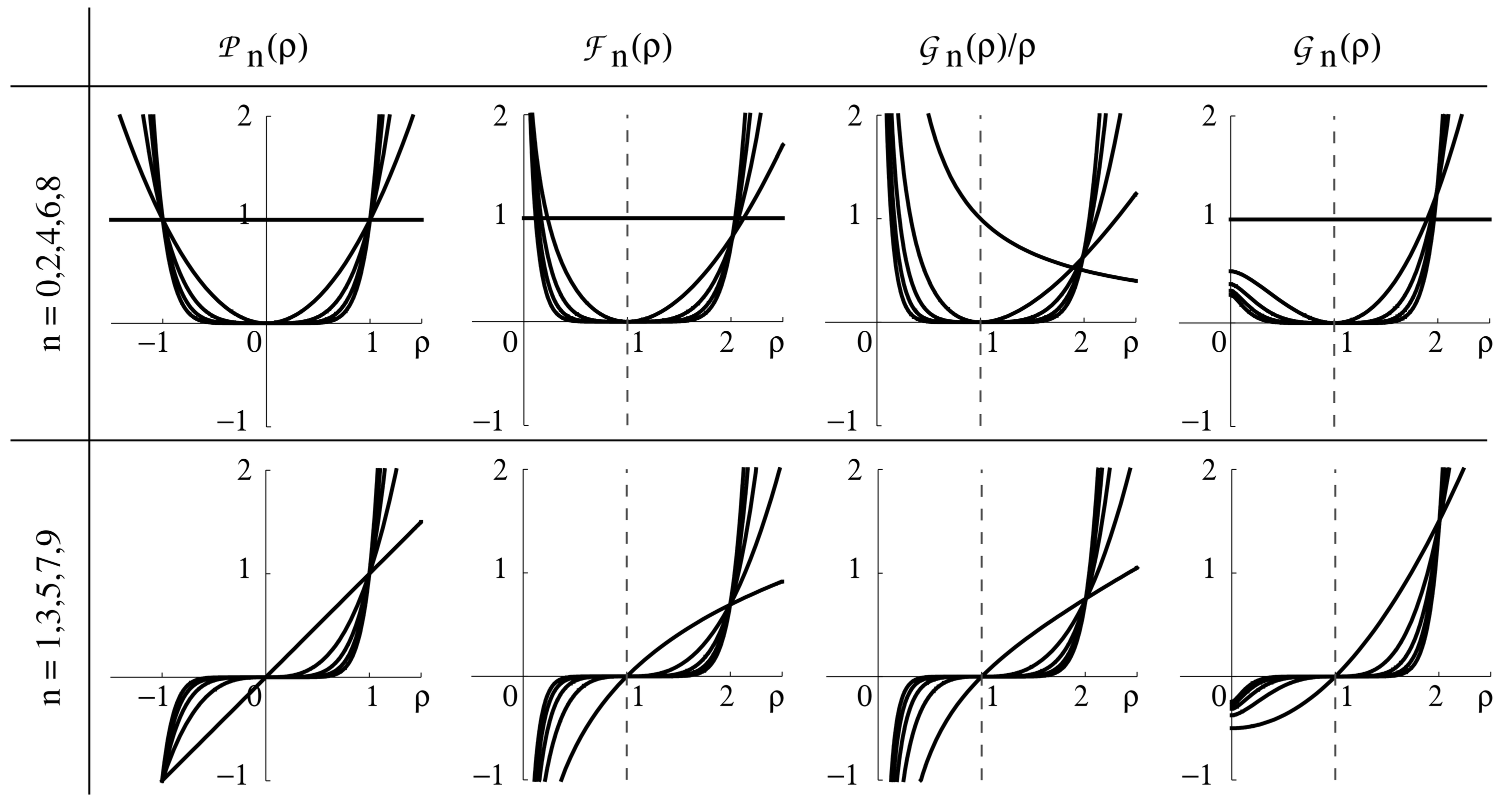
with lowering operator which relates functions of different orders to each other

$$n\,(\mathcal{A},\mathcal{B})_{n-1} = \frac{\partial}{\partial x}(\mathcal{A},\mathcal{B})_n = \pm\frac{\partial}{\partial y}(\mathcal{B},\mathcal{A})_n \qquad \mathcal{A}_0 = 1,\ \mathcal{B}_0 = 0.$$

Normal		Skew	
$\overline{\Phi}^{(n)} = -\mathrm{C}_n\frac{\mathcal{B}_n^{(\mathrm{e})}}{n!}$	$\overline{A}_\theta^{(n)} = -\mathrm{C}_n\frac{\mathcal{A}_n^{(\mathrm{m})}}{n!}$	$\underline{\Phi}^{(n)} = -\mathrm{D}_n\frac{\mathcal{A}_n^{(\mathrm{e})}}{n!}$	$\underline{A}_\theta^{(n)} = \mathrm{D}_n\frac{\mathcal{B}_n^{(\mathrm{m})}}{n!}$
$\overline{F}_\rho^{(n)} = \mathrm{C}_n\frac{\mathcal{B}_{n-1}^{(\mathrm{m})}}{(n-1)!}$	$\overline{F}_{\mathrm{y}}^{(n)} = \mathrm{C}_n\frac{\mathcal{A}_{n-1}^{(\mathrm{e})}}{(n-1)!}$	$\underline{F}_\rho^{(n)} = \mathrm{D}_n\frac{\mathcal{A}_{n-1}^{(\mathrm{m})}}{(n-1)!}$	$\underline{F}_{\mathrm{y}}^{(n)} = -\mathrm{D}_n\frac{\mathcal{B}_{n-1}^{(\mathrm{e})}}{(n-1)!}$



3. Multipoles in Cylindrical Coordinates and McMillan Harmonics



First step is to restore the symmetry

$$\triangle_\frown \Phi = \triangle_\perp \Phi + \frac{1}{\rho}\frac{\partial\,\Phi}{\partial\rho} = \frac{\partial^2\Phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\,\Phi}{\partial\rho} + \frac{\partial^2\Phi}{\partial y^2} = 0$$

$$\triangle_\frown \mathbf{A} = \left(\triangle_\frown A_\theta - \frac{A_\theta}{\rho^2}\right)\hat{\mathbf{e}}_\theta = \frac{\hat{\mathbf{e}}_\theta}{\rho}\left[\frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial y^2}\right](\rho\,A_\theta) = 0$$

Looking for the solution in a form similar to HHP

$$\Phi = -\sum_{k=0}^{n}\frac{\mathcal{F}_{n-k}(\rho)}{(n-k)!}\frac{y^k}{k!}\left(\mathrm{C}_n\,\sin\frac{k\,\pi}{2} + \mathrm{D}_n\,\cos\frac{k\,\pi}{2}\right)$$

$$A_\theta = -\sum_{k=0}^{n}\frac{1}{\rho}\frac{\mathcal{G}_{n-k}(\rho)}{(n-k)!}\frac{y^k}{k!}\left(\mathrm{C}_n\,\cos\frac{k\,\pi}{2} - \mathrm{D}_n\,\sin\frac{k\,\pi}{2}\right)$$

where functions $\mathcal{F}_n(\rho)$ and $\mathcal{G}_n(\rho)$ are determined by two recurrence equations

$$\frac{\partial^2\mathcal{F}_n(\rho)}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\,\mathcal{F}_n(\rho)}{\partial\rho} = n\,(n-1)\,\mathcal{F}_{n-2}(\rho)$$

$$\frac{\partial^2\mathcal{G}_n(\rho)}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial\,\mathcal{G}_n(\rho)}{\partial\rho} = n\,(n-1)\,\mathcal{G}_{n-2}(\rho)$$

one can see that $\mathcal{F}_n$ and $\mathcal{G}_n$ are related to each other through

$$\mathcal{G}_{n-1} = \frac{1}{n}\rho\frac{\partial\,\mathcal{F}_n}{\partial\rho} \quad \text{and} \quad \mathcal{F}_{n-1} = \frac{1}{n}\frac{1}{\rho}\frac{\partial\,\mathcal{G}_n}{\partial\rho}$$

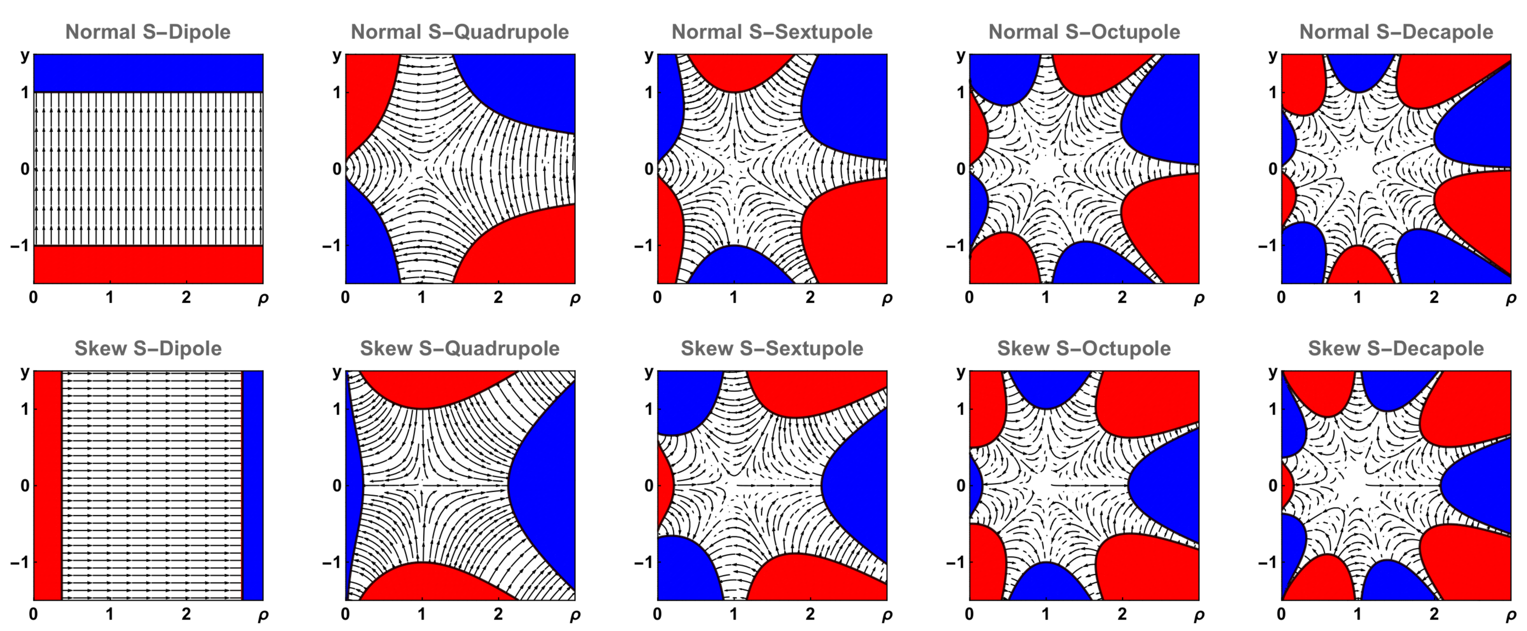
and allow the construction of lowering and corresponding raising operators

$$\mathcal{F}_n = \frac{1}{(n+1)(n+2)}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)\right]\mathcal{F}_{n+2} \qquad \mathcal{F}_n = n\,(n-1)\int_1^\rho\frac{1}{\rho}\int_1^\rho \rho\,\mathcal{F}_{n-2}\,\mathrm{d}\,\rho\,\mathrm{d}\,\rho$$

$$\mathcal{G}_n = \frac{1}{(n+1)(n+2)}\left[\rho\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\right]\mathcal{G}_{n+2} \qquad \mathcal{G}_n = n\,(n-1)\int_1^\rho\rho\int_1^\rho \frac{1}{\rho}\mathcal{G}_{n-2}\,\mathrm{d}\,\rho\,\mathrm{d}\,\rho$$

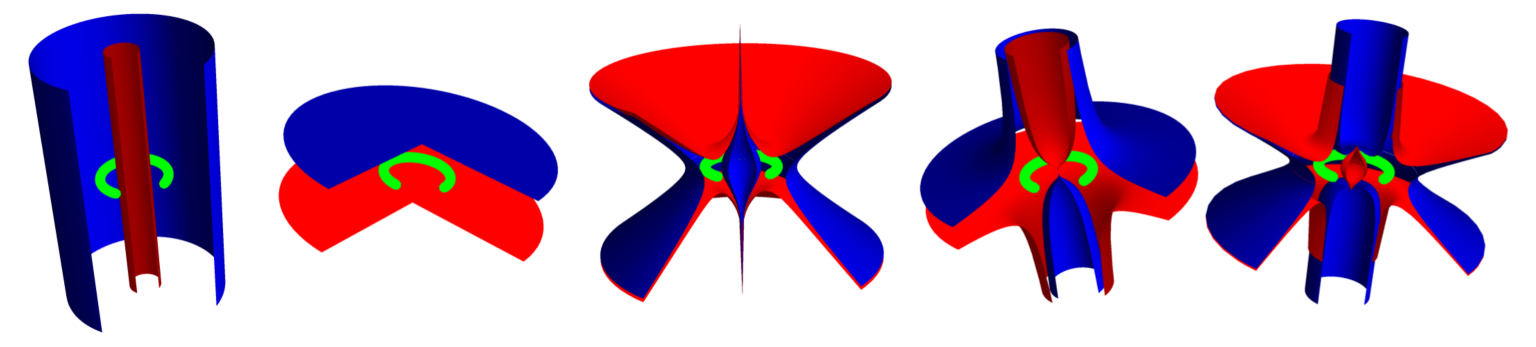
with an additional constraint to terminate recurrences defining lowest orders as

$$\mathcal{F}_0 = 1, \qquad \mathcal{F}_1 = \ln\,\rho, \qquad \mathcal{G}_0 = 1, \qquad \mathcal{G}_1 = (\rho^2-1)/2.$$



4. Summary

https://arxiv.org/pdf/1603.03451v1.pdf



3D models of 2n-pole sector magnets. From the left to the right: skew S-dipole, normal S-dipole, skew S-quadrupole, normal S-quadrupole and skew S-sextupole.

Table 1: **Sector harmonics.**

	n	
$\mathcal{A}_n^{(\mathrm{e})}$	0	1
	1	$\ln\rho$
	2	$\left[\frac{\rho^2-1}{2} - y^2\right] - \ln\rho$
	3	$\left[-3\,\frac{\rho^2-1}{2}\right] + 3\,\left(\frac{\rho^2+1}{2} - y^2\right)\ln\rho$
	4	$\left[\frac{3(\rho^4+4\,\rho^2-5)}{8} - 6\,\frac{\rho^2-1}{2}\,y^2 + y^4\right] - 3\,\left(\frac{1}{2} + \rho^2 - 2\,y^2\right)\,\ln\rho$
$\mathcal{A}_n^{(\mathrm{m})}$	0	$\frac{1}{\rho}$
	1	$\frac{1}{\rho}\left[\frac{\rho^2-1}{2}\right]$
	2	$\frac{1}{\rho}\left\{\left[-\frac{\rho^2-1}{2} - y^2\right] + \rho^2\ln\rho\right\}$
	3	$\frac{1}{\rho}\left\{\left[\frac{3(\rho^2+1)}{4}\frac{\rho^2-1}{2} - 3\,\frac{\rho^2-1}{2}\,y^2\right] - \frac{3}{2}\,\rho^2\ln\rho\right\}$
	4	$\frac{1}{\rho}\left\{\left[-\frac{3(5\,\rho^4-4\,\rho^2-1)}{8} + 6\,\frac{\rho^2-1}{2}\,y^2 + y^4\right] + \frac{3(2+\rho^2-4\,y^2)}{2}\,\rho^2\ln\rho\right\}$
$\mathcal{B}_n^{(\mathrm{e})}$	0	0
	1	y
	2	$y\,[2\,\ln\rho]$
	3	$y\left\{\left[3\,\frac{\rho^2-1}{2} - y^2\right] - 3\,\ln\rho\right\}$
	4	$y\left\{\left[-12\,\frac{\rho^2-1}{2}\right] + 4\,\left(3\,\frac{\rho^2+1}{2} - y^2\right)\ln\rho\right\}$
$\mathcal{B}_n^{(\mathrm{m})}$	0	0
	1	$\frac{y}{\rho}$
	2	$\frac{y}{\rho}\left[2\,\frac{\rho^2-1}{2}\right]$
	3	$\frac{y}{\rho}\left\{\left[-3\,\frac{\rho^2-1}{2} - y^2\right] + 3\,\rho^2\ln\rho\right\}$
	4	$\frac{y}{\rho}\left\{\left[4\,\frac{3(\rho^2+1)}{4}\frac{\rho^2-1}{2} - 4\,\frac{\rho^2-1}{2}\,y^2\right] - 6\,\rho^2\ln\rho\right\}$

Table 2: **Relationship between the coefficients of pure normal and skew sector multipoles, and, power series expansion of field in radial and vertical planes.**

	C_n		D_n	
n	$x=0$	$y=0$	$x=0$	$y=0$
1	F_y	F_y	F_x	F_x
2	$\partial_y\,F_x$	$\partial_x\,F_y$	$-\partial_y\,F_y$	$\partial_x\,F_x + F_x$
3	$-\partial_y^2\,F_y$	$\partial_x^2\,F_y + \partial_x\,F_y$	$-\partial_y^2\,F_x$	$\partial_x^2\,F_x + \partial_x\,F_x - F_x$
4	$-\partial_y^3\,F_x$	$\partial_x^3\,F_y + \partial_x^2\,F_y - \partial_x\,F_y$	$\partial_y^3\,F_y$	$\partial_x^3\,F_x + 2\,\partial_x^2\,F_x - \partial_x\,F_x + F_x$