SYNCROTRON OSCILLATION DERIVED FROM THREE COMPONENTS HAMILTONIAN

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Abstract

The Hamiltonian, which was composed of coasting, synchrotron and betatron motions, clarified the synchro-betatron resonant coupling mechanism in a storage ring. The equation for the synchrotron motion was also obtained from the Hamiltonian. It shows that the so-called synchrotron oscillation is an oscillation around the revolution frequency as well as of the kinetic energy of the on-momentum particle. The detectable synchrotron oscillation is a horizontal oscillation on the laboratory frame.

INTRODUCTION

The Hamiltonian, which was composed of coasting, synchrotron and betatron motions, was derived to explain the observed horizontal betatron tune jump near the synchro-betatron resonant coupling point[1]. We discuss about the so-called synchrotron oscillation related to the Doppler effect. We show that the synchrotron oscillation derived from the Hamiltonian is longitudinal as well as horizontal oscillations.

HAMILTONIAN FOR THE ORBITING PARTICLE

We assume that an on-momentum particle of mass m and momentum p_0 is circling (without oscillating motion) the reference closed orbit of the average radius R with velocity v. E_0 is the total energy and K is the kinematic energy of the on-momentum particle. We have the following relations:

$E_0{}^2 = (p_0 c)^2 + (mc^2)^2$, $p_0 = m\gamma\beta c$, and $E_0 = K + mc^2$,

where $\beta = \frac{v}{c}$, $\frac{ds}{dt} = v$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $E_0 = m\gamma c^2$.

Keeping up to the 2nd order to describe an orbiting particle of coasting, betatron and synchrotron motions, the Hamiltonian composed of three motions is given as follows from Eq.(21) of the reference 1:

$$\begin{aligned}\bar{H} &= -(1+\delta_C+\delta_S) + \frac{1}{2}\left(\frac{\bar{p}_x}{p_0}\right)^2 + \frac{1}{2}K_x\bar{x}^2 + \frac{1}{2}(-\eta)(\delta_C+\delta_S)^2 \\ &-\frac{hqV}{2\pi\beta^2 E_0}\left\{\cos(\phi+\phi_D) - \cos(\phi_S+\phi_D) + (\phi-\phi_S)\sin(\phi_S+\phi_D)\right\}\end{aligned} \tag{1}$$

where δ_S is the (rationalized) fractional deviation of the kinetic energy caused by the so-called synchrotron oscillation and δ_C is the (rationalized) fractional deviation of the kinetic energy caused by the dispersion. Both δ_S and δ_C are normalized by $\beta^2 E_0$. The coasting motion consists of the 0th (on-momentum particle) and 1st (δ_S and δ_C)order effects. η is the phase slip factor. We assume the strong focusing case $0 < (-\eta) < 1$. ϕ is the phase of the rf wave. Around the off-momentum closed orbit, $\bar{x}$ is horizontal coordinate and $\bar{p}_x$ is horizontal momentum of the orbiting particle. We also have $\phi_D = -\frac{D}{R}\left(\frac{\bar{p}_x}{p_0}\right) + \frac{D'}{R}\bar{x}$, and $\phi_S = \psi_S - \phi_D$. D is the dispersion and ψ_S is the phase angle for synchronous particle with respect to the rf cavity voltage V. In the following argument, we neglect the betatron oscillation.

OSCILLATION AROUND THE REVOLUTION FREQENCY

We obtain the following equation for the so-called synchrotron oscillation from Eqs.(26) and (27) of the reference 1:

$$\delta_S = \hat{\delta}\cos\left\{\nu_\eta(\theta-\theta_0)\right\} + \delta_0 + C \tag{2}$$

$$\nu_\eta^2 = \frac{\omega_S^2}{\omega_0^2} = \frac{hqV\left|\eta\cos\left(\phi_S+\phi_D\right)\right|}{2\pi\beta^2 E_0}. \tag{3}$$

where $\delta_0 = -\delta_C - \frac{1}{\eta}$, θ is the orbit angle ($\theta = \frac{s}{R}$), θ_0 is the orbit angle at the injection point. ω_0 is the revolution frequency, ω_s is the synchrotron frequency, ν_η is the synchrotron tune, $\hat{\delta}$ is the amplitude of oscillation and C is an integration constant.

Generally $\delta_C << 1$ and we can choose $\delta_C = \hat{\delta}$ at $\theta = \theta_0$. We can neglect this term. Then we consider only the reference closed orbit. Now the coasting motion consists of the on-momentum particle plus the effect of δ_S. From Eq.(2)

$$\delta_S = \hat{\delta}\cos\left\{\nu_\eta(\theta-\theta_0)\right\} + \frac{1}{(-\eta)} + C \tag{4}$$

Neglecting the integration constant $C = 0$,

$$-\eta\delta_S = -\eta\hat{\delta}\cos\left\{\nu_\eta(\theta-\theta_0)\right\} + 1. \tag{5}$$

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We have the following relation[2]:

$$\frac{\Delta\omega}{\omega_0} = -\eta\delta_S . \qquad (6)$$

Define $\frac{\Delta\hat{\omega}}{\omega_0} = -\eta\hat{\delta}$. Then we have

$$\frac{\Delta\omega}{\omega_0} = \frac{\Delta\hat{\omega}}{\omega_0}\cos\left\{\nu_\eta\left(\theta-\theta_0\right)\right\}+1 . \qquad (7)$$

The last term of RHS of Eq.(7) represents the particle circling with ω_0 . This term shows up because of the 1st order term of the coasting motion δ_S in Eq.(1). The synchrotron oscillation, which is an oscillation around the revolution frequency ω_0 , occurs on the mechanical frame of the circling particle. Then detected synchrotron tunes are expected to be changed on the laboratory frame by the Doppler effect.

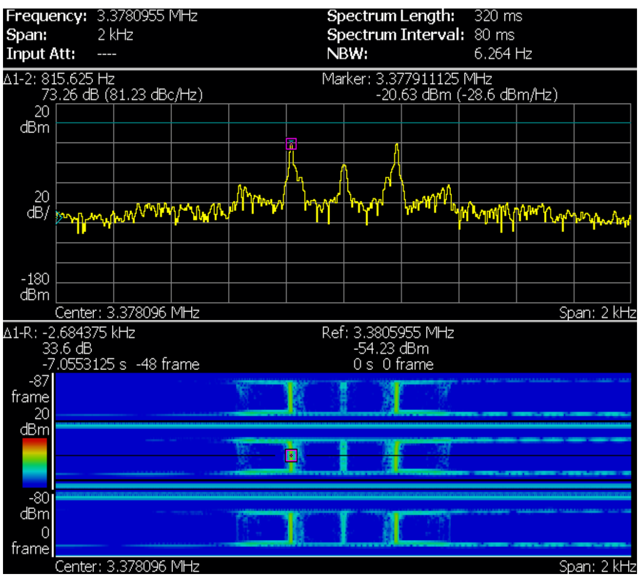

Figure 1: An image of the spectrum analyzer taken for $^{12}C^{6+}$ (400MeV/u) beam of a heavy ion synchrotron. Two symmetrical sidebands (±0.185kHz) around the revolution frequency (3.378096MHz) was clearly recorded.

EXPERIMENTAL RESULTS

For low energy particles (non-relativistic), recorded synchrotron frequencies $\omega^{\pm}$ are found symmetrically as two sidebands around the revolution frequency as follows

$$\omega^{\pm}_{non\ relativistic} = \omega_0 \pm \omega_s , \qquad (8)$$

where $\omega^{+}_{non\ relativistic}$ for the oscillation moving toward the observer and $\omega^{-}_{non\ relativistic}$ for the oscillation moving away from the observer. For relativistic particles, however, the synchrotron frequencies should be found non-symmetrically around the revolution frequency by the Doppler effect as follows[3]:

$$\omega^{+}_{relativistic} = \omega_0 + \omega_s\gamma\left(1+\beta\right) , \qquad (9\text{-a})$$

$$\omega^{-}_{relativistic} = \omega_0 - \omega_s\gamma\left(1-\beta\right) . \qquad (9\text{-b})$$

where $\omega^{+}_{relativistic}$ for the oscillation moving toward the observer and $\omega^{-}_{relativistic}$ for the oscillation moving away from the observer.

Figure 1 is an image of the spectrum analyzer (Tektronix RSA 3303B) taken for $^{12}C^{6+}$ (400MeV/u) beam of a heavy ion synchrotron at Gunma university heavy ion medical centre[4]. Two symmetrical sidebands (±0.185kHz) around the revolution frequency (3.378096MHz) was clearly recorded. The image was taken by the electrostatic beam positioning monitor, of which plate was used as an antenna. The antenna acted as the observer. The detected synchrotron tune was 0.000055. No Doppler effect was confirmed.

PHENOMENOLOGICAL ANALYSIS

Since no Doppler effect was confirmed, the synchrotron oscillation occurs not on the frame of the circling particle, but on the laboratory frame. Eq.(7) should turn to be

$$\Delta\omega = \Delta\hat{\omega}\cos\left\{\nu_\eta\left(\theta-\theta_0\right)\right\} . \qquad (10)$$

However, small frequency oscillation without circling motion is meaningless as the synchrotron oscillation. To explain physical reason why no Doppler was confirmed, we go back to Eq.(4).

Following the definition of (rationalized) fractional deviation δ of reference 1, we define as follows:

$$\delta_S = \frac{\Delta E_S}{\beta^2 E_0} , \ \hat{\delta} = \frac{\Delta\hat{E}}{\beta^2 E_0} \ \text{and} \ \delta_\eta = \frac{E_\eta}{\beta^2 E_0} = \frac{1}{(-\eta)} .$$

Neglecting the integration constant $C = 0$, we rewrite Eq.(4) as follows,

$$\Delta E_S = \Delta\hat{E}\sin\left\{\nu_\eta\left(\theta-\theta_0\right)\right\} + E_\eta . \qquad (11)$$

where ΔE_S is the kinetic energy of the oscillation and $\Delta\hat{E}$ is its amplitude.

In Eq. (11), E_η is the centre energy of the oscillation. We call E_η the oscillation centre. To find out a role of the oscillation centre, we neglect oscillating motions as 1st order effect and consider only motion of the oscillation centre. Let's locates a non-oscillating imaginary particle of mass m at the oscillation centre. E_η represents the kinetic energy of the imaginary particle against the laboratory frame. We evaluate the fictitious velocity u_C of the oscillation centre. Define the momentum p_i and the total energy E_i of an imaginary particle as follows:

$$E_i^2 = \left(p_i c\right)^2 + \left(mc^2\right)^2 , p_i = m\gamma_C u_C \ \text{and} \ E_i = E_\eta + mc^2 .$$

We have

$$\gamma_C = \frac{E_i}{mc^2} = \frac{\frac{1}{(-\eta)}\beta^2 E_0 + mc^2}{mc^2} = 1 + \frac{\beta^2\gamma}{(-\eta)}. \quad (12)$$

When the oscillation centre is on the on-momentum particle ($\gamma = \gamma_C$), we define $\eta = \eta_C$.

From Eq.(12), we have

$$-\eta_C = \frac{\beta^2}{1-\left(\frac{1}{\gamma}\right)} = 1 + \frac{1}{\gamma}. \quad (13)$$

For $\eta < 0$, following relations are obtained:

1) $-\eta \to 0$, the oscillation centre moves with $u_C \to c$.

2) $0 < -\eta < 1 + \frac{1}{\gamma}$, the oscillation centre moves faster than the on-momentum particle.

3) $-\eta = -\eta_C = 1 + \frac{1}{\gamma}$, the oscillation centre is on the on-momentum particle.

4) $1 + \frac{1}{\gamma} < -\eta$, the oscillation centre moves slower than the on-momentum particle.

5) $-\eta \to \infty$, the oscillation centre locates on the lab frame $(u_C = 0)$.

Generally it is very difficult to find appropriate values of η (assuming $0 < (-\eta) < 1$), which does not satisfy conditions 1), 3), 4) and 5). The condition 2) should be satisfied, but it says that the velocity of the oscillation centre is faster than the velocity of the on-momentum particle. This result is unrealistic.

Since Eq.(11) is equivalent to Eq.(7), the imaginary particle, which circles with ω_0, is on the kinetic frame of the on-momentum particle. Accordingly the imaginary particle, which locates on the oscillation centre, should corresponds to the on-momentum particle.

We had to choose an appropriate integration constant $C \neq 0$ in Eq.(4) and redefine E_η as follows:

$$\frac{E_\eta}{\beta^2 E_0} = \frac{1}{(-\eta)} + C \quad (14)$$

As E_η equals to the kinetic energy of $K = (\gamma - 1)mc^2$,

$$C = \frac{1}{\left(1+\frac{1}{\gamma}\right)} - \frac{1}{(-\eta)}. \quad (15)$$

CONSIDERATION

The so-called synchrotron oscillation is the oscillation of the revolution frequency and the oscillation of the kinetic energy of the on-momentum particle. Two pictures are equivalent but represent oscillations of two different directions. Since the first one occurs in the orbital direction, it is an oscillation in the longitudinal direction. We call it the longitudinal oscillation. We define the longitudinal constant $C \to C_\ell = 0$, from Eq.(7)

$$\Delta\omega = \Delta\hat{\omega}\cos\left\{\nu_\eta\left(\theta - \theta_0\right)\right\} + \omega_0, \quad (16)$$

where $\Delta\omega$ is a small value on the mechanical frame of the circling particle.

For the second one, the radius of the circling particle also oscillates around the reference closed orbit when its kinetic energy oscillates. It is an oscillation in the horizontal direction. We call it the horizontal oscillation. We define the horizontal constant $C \to C_h = \frac{1}{\left(1+\frac{1}{\gamma}\right)} - \frac{1}{(-\eta)}$, from Eq.(11)

$$\Delta E_S = \Delta\hat{E}\sin\left\{\nu_\eta\left(\theta - \theta_0\right)\right\} + (\gamma - 1)mc^2, \quad (17)$$

where ΔE_S is a small value on the mechanical frame of the on-momentum particle. C_h erases $\frac{1}{(-\eta)}$ term in Eq.(4). Now η influences directory the oscillation through Eq.(3) but does not influence indirectly the oscillation through the mechanical frame.

In the laboratory frame, the velocity of the circling particle is changed periodically but very slowly in the longitudinal oscillation. In practical situation, however, the particle revolves many times (more than 10^4 times/sec for Fig. 1 case) for one longitudinal oscillation and no synchrotron tune in that direction is detectable in the laboratory frame. It is reasonable that the Doppler effect in the longitudinal direction was not recorded in Fig. 1. This result also demonstrates that the horizontal oscillation is not moving with the circling particle. It occurs on the laboratory frame. We conclude that the synchrotron oscillation is a horizontal oscillation on the laboratory frame and the synchrotron tune is detectable only in the horizontal direction.

One of the author reported that energy exchange between the synchrotron and the betatron oscillations was possible[1]. The betatron oscillation is also a horizontal oscillation. It is well know that an electrostatic kicker is utilized to excite betatron oscillations. Therefore, on the same way, we may utilize a kicker to de-excite synchrotron oscillations slowly so that particles outside the bucket are pushed back into the bucket to reduce loosing particles.

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