# INTRINSIC LANDAU DAMPING OF SPACE CHARGE MODES AT COUPLING RESONANCE*

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*Abstract*

Using Synergia accelerator modeling package and Dynamic Mode Decomposition technique, the properties of the first transverse dipole mode in Gaussian bunches with space charge are compared at transverse coupling resonance and off-resonance. The Landau damping at coupling resonance and in the strong space charge regime is a factor of two larger, while the mode's tune and shape are nearly the same. While the damping mechanism in the off-resonance case fits well with the classical Landau damping paradigm, the enhancement at coupling resonance is due to a higher order mode-particle coupling term which is modulated by the amplitude oscillation of the resonance trapped particles.

## INTRODUCTION

Landau damping (LD) mechanism [1] is an important research topic in plasma and accelerator physics. The damping is caused by the energy exchange between a coherent mode and the particles in resonance with the mode. In the typical picture, LD requires the coherent resonance line to lie within the incoherent spectrum. Our numerical investigation of the transverse space charge (SC) modes in bunched beams reveals a novel damping mechanism at the coupling resonance (CR), *i.e.*, when the horizontal and the vertical tunes are close. In contrast with the usual LD mechanism, the tunes of the LD-responsible particles, *i.e.*, the particles which absorb the mode energy, have a wide spread. This happens due to the oscillatory behavior of the amplitudes of the CR trapped particles, which modulates the mode-particle coupling.

The transverse SC modes in bunched beams away from the CR were calculated in Refs [2–4]. Their intrinsic LD in the strong SC regime was suggested in Ref [2, 3]. Predicted damping rates were confirmed by numerical simulations [5–7]. The linear CR influence on LD was addressed in Ref [8].

In our study there is no linear coupling term between the transverse planes since the SC force introduces only higher-order coupling terms. The main resonance is the fourth-order Montague resonance [9] resulting from the term proportional to $x^2y^2$ in the SC potential. The particles trapped in the resonance islands are characterized by an oscillatory energy exchange between the transverse planes. Their transverse amplitudes are oscillating with typical trapping frequencies. The mode-particle coupling is therefore modulated by the trapping frequency since it is dependent on the particle's amplitudes. Because the trapping frequencies are particle dependent, the tunes of the LD-responsible particles are particle dependent also.

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We compare the properties of the first SC mode both on and off the CR. We find that the LD is larger in the former case. By investigating the properties of the particles exchanging energy with the mode, we conclude that the off-resonance case well fits the conventional LD scenario characterized by LD-responsible particles with an incoherent tune spectrum at the coherent tune. At CR the damping enhancement is caused by the presence of the modulated coupling between the mode and the trapped particles. Our approach does not assume any analytical model; it is solely based on numerical simulations of a bunch propagating through a lattice.

## FORMALISM

The mode-particle interaction equation can be written [10]

$$\ddot{x} + \omega_0^2 (Q_{0x} - \delta Q)^2 x = -2\omega_0^2 Q_{0x} \delta Q \bar{x} \quad (1)$$

The SC mode enters in Eqs. 1 and 2 as $\bar{x}(t) = e^{-i\omega_0 \nu t}\bar{x}[z(t)]$, where $\nu$ is the mode tune. For the first mode, $\bar{x}[z] \approx \sin[\pi z/4\sigma_z]$ [2, 3]. Taking into account the synchrotron oscillations of $z(t)$, the Bessel function expansion of $\bar{x}(t)$ yields the mode-particle main resonant exchange tune at $\nu - Q_s$, where $Q_s$ is the synchrotron tune. The tune shift $\delta Q(z, J_x, J_y)$ is proportional to the line charge density and is dependent on the particle transverse actions. To a good approximation, $J_x$ and $J_y$ are constants of motion. The resonant energy exchange between the mode and the particle occurs when the particle tune is close to $\nu - Q_s$.

The situation is different at the CR. In the proximity of the resonance, the sum $J_s = J_x + J_y$ is a constant of motion, while the difference $J_d = J_x - J_y$ oscillates around the stable point. Using the $J_d$ expansion of the mode-particle coupling term, $2\omega_0^2 Q_{0x}\delta Q = A + BJ_d$, Eq. 1 can be written as

$$\ddot{x} + \omega_0^2 Q_x(z, J_s, J_d)^2 x = -A(z, J_s)\bar{x} - B(z, J_s)J_d\bar{x}. \quad (2)$$

The trapping frequency $\omega_0 Q_t$, *i.e.*, the frequency of the $J_d$ oscillations at CR, is particle dependent [9]. The oscillations of $J_d$ contribute to the damping in two ways. First, the dependence of $Q_x(z, J_s, J_d)$ on $J_d$ yields satellites spaced by harmonics of $Q_t$ in the incoherent spectrum. These satellites are resonant with the particle-mode coupling term $A\bar{x}$ when their tune is at $\nu - Q_s$. Second, the $J_d$ oscillations modulate the particle-mode coupling term $BJ_d\bar{x}$, yielding a novel damping mechanism. If in the conventional picture, the LD requires particles with an incoherent spectrum covering the mode frequency, *i.e.*, $\bar{Q}_x \approx \nu - Q_s$, where $\bar{Q}_x$ is the particle's main tune, the $BJ_d\bar{x}$ term implies mode-resonant particles when $\bar{Q}_x \approx \nu - Q_s - Q_t$. Because $Q_t$ is particle dependent, $\bar{Q}_x$ of the particles participating to the parametric LD is particle dependent, too, and may spread over a large range.

The simulations are done by employing the particle tracking code Synergia [11, 12]. The SC effects are implemented using the second order split-operator method [13]. The electric field is calculated by numerically solving the 3D Poisson equation with open boundary conditions [14]. The bunch is initially excited in the horizontal plane with the first SC harmonic function. The transverse displacement density, $X(z, \delta p/p, t)$ is calculated at every turn. The modes' shape, tune and damping are extracted from $X(z, \delta p/p, t)$ using the DMD technique [15–21]. Application of Synergia and DMD to beam dynamics is described in detail in [7].

A lattice made by 10 identical OFORODO (drift - focusing quad - drift - rf cavity - drift - defocusing quad - drift) cells is chosen. $10^8$ macroparticles per bunch are used for the simulations. For the off-resonance case we take the bare betatron tune difference $Q_{0x} - Q_{0y} > \delta Q_{sc\ max}$ while at the CR $Q_{0x} = Q_{0y}$. $\delta Q_{sc\ max}$ is the SC tune shift at the center of the bunch. The chromaticity is zero. The beam distribution is longitudinally and transversely Gaussian with equal vertical and horizontal emittances. The SC parameter is defined as $q = \frac{\delta Q_{sc\ max}}{Q_s}$.

## RESULTS

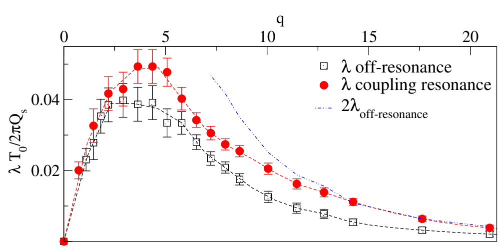


Figure 1: Landau damping rate ($\lambda$) vs. space charge parameter ($q$) on and off the CR. LD is a factor of 2 larger at CR when SC is large.

The damping rate of the first SC mode off-resonance and at CR is compared in Fig. 1. For intermediate and large SC, $q \gtrsim 4$, the damping at CR is larger. In the strong SC regime, $10 \lesssim q \lesssim 20$, the damping at CR is larger by approximately a factor of 2. One the other hand, the mode's tune and shape are nearly the same in both cases (not shown).

The off-resonance LD mechanism can be understood within the typical paradigm. The beam 2D tune footprint, $\rho(Q_x, Q_y)$, is plotted in Fig. 2-a. $\rho(Q_x, Q_y)$ is defined as

$$\rho(Q_x, Q_y) = \sum_i |\tilde{x}_i(Q_x)|^2 |\tilde{y}_i(Q_y)|^2, \qquad (3)$$

where $\tilde{x}_i(Q)$ ($\tilde{y}_i(Q)$) is the Fourier transform of the particle $i$ horizontal (vertical) displacement $x_i(t)$ ($y_i(t)$) normalized to one. The SC force shifts the particles' tunes to lower values. The satellite lines separated by $2Q_s$ are a consequence of the modulation of the tune shift with the particle's longitudinal position. The particles directly responsible for the LD are the ones which resonantly exchange energy with the mode. To select the LD-responsible particles we look for those having the largest change in their energy during the simulation. In Fig. 2-b (-c) we plot the sum of $\Delta J_x = J_x - J_{x\ initial}$ and the sum of $\Delta J_y = J_y - J_{y\ initial}$ for the 0.05% and the 0.2% largest energy increase (decrease) particles. 0.05% and 0.2% are arbitrary chosen values for the purpose of illustrating the properties of the LD-responsible particles. Note that the chosen particles increase or decrease their action only in the horizontal plane, *i.e.*, the plane where the mode is present. The 2D footprint of the 0.5% largest-energy-changing particles is shown in Fig. 2-d. As expected, since these particles are mode-resonant, their horizontal tune is in the vicinity of $\nu - Q_s$.

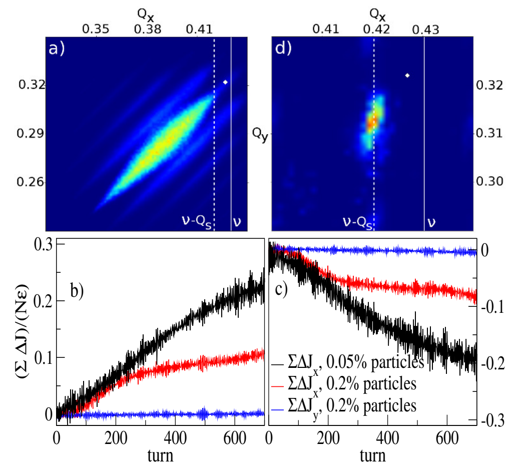


Figure 2: a) Bunch tune footprint at off-resonance, $q = 7.94$. The white dot corresponds to the bare betatron tunes. b) $\sum_i \Delta J_x = \sum_i (J_{xi} - J_{xi\ initial})$ and $\sum_i \Delta J_{yi} = \sum_i (J_{yi} - J_{yi\ initial})$ of the 0.05% and 0.2% largest increasing energy particles versus turn number, normalized by the product of emittance and the number of the particles in the sum. c) The same as b) but for the largest decreasing energy particles. d) Tune footprint for the 0.5% largest changing energy (increase and decrease) particles. The tunes are in the proximity of the coherent tune $Q_x = \nu - Q_s$.

The spectral properties of the LD-responsible particles at the CR do not fit the typical LD paradigm. The beam 2D tune footprint in Fig. 3-a displays enhanced spectral weight along the coupling resonance line $2Q_x - 2Q_y = 0$, consequence of resonance trapping. $2Q_s$-spaced satellite lines can be observed. We use the same largest energy change criterion to select the LD-responsible particles. Unlike the off-resonance case, the horizontal and vertical actions exhibit non-monotonic change with turn number, since in the proximity of CR their magnitude oscillates between the planes. However, the transverse action sum $J_s$ of the LD-responsible particles displays a monotonic increase (decrease), as shown in Fig 3-b(-c). The interesting fact which points to an unconventional damping mechanism is that the tune of most of these large energy changing particles is not in the vicinity of $\nu - Q_s$ as one would expect for LD-responsible particles. As shown in Fig 3-d, there is a large spectral weight on the CR line which extends well below $Q_x = \nu - Q_s$.

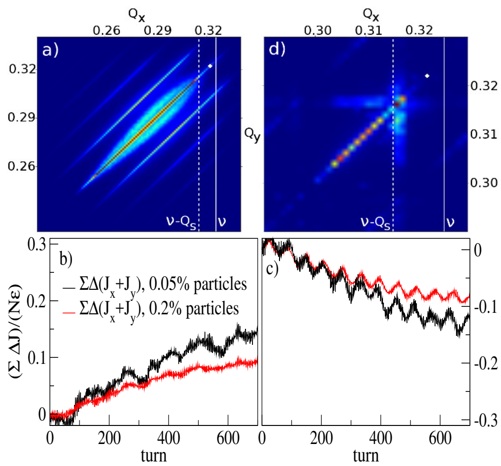


Figure 3: a) Bunch tune footprint at CR, $q = 7.94$. The white dot corresponds to the bare betatron tunes. b) $\sum_i \Delta J_{si} = \sum_i (J_{si} - J_{si\ initial})$ of the 0.05% and 0.2% largest increasing energy particles versus turn number, normalized by the product of emittance and the number of the particles in the sum. c) The same as b) but for the largest decreasing energy particles. d) Tune footprint for the 0.5% largest changing energy (increase and decrease) particles. Large part of the spectral weight is along the resonance line $2Q_x - 2Q_y = 0$, with the horizontal tune well below $\nu - Q_s$.

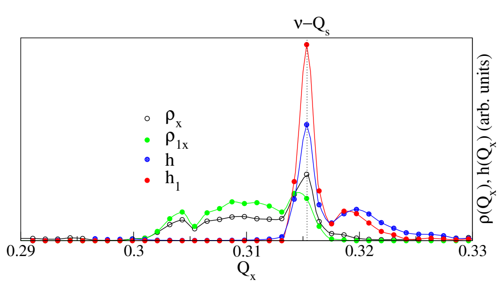


Figure 4: Coupling resonance, $q = 7.94$. The horizontal tune density $\rho(Q_x)$ (black), the $Q_t$ shifted tune density $h(Q_x)$ (blue) and the one-tune-per-particle tune density $\rho_1(Q_x)$ (green) with the corresponding $Q_t$ shifted tune density $h_1(Q_x)$ for the 0.5% largest changing energy particles. $h(Q_x)$ and $h_1(Q_x)$ are strongly peaked at the resonant mode tune $Q_x = \nu - Q_s$, showing mode-particle resonance via the $BJ_d\bar{x}$ term.

Most of the large changing energy particles are trapped in resonance islands. The $Q_t$ satellites in the particles' tune spectra contribute to the spectral weight at the mode coherent tune by $\approx 20\% \sim 25\%$. To estimate the satellites' spectral weight we compare the horizontal tune density $\rho_x(Q) = \sum_i \rho_{xi}(Q)$ and the one-tune-per-particle density $\rho_{1x}(Q) = \sum_i \rho_{1xi}(Q)$. The sum here is restricted only to the number of the selected particles with the largest energy change. $\rho_{xi}(Q) = |\tilde{x}_i(Q)|^2$ and $\rho_{1xi}(Q) = \delta(Q - \bar{Q}_{xi})$. $\bar{Q}_{xi}$ is the tune of the largest spectral peak in the Fourier spectrum $|\tilde{x}_i(Q)|$. Unlike $\rho_x$, where all spectral features are present, $\rho_{1x}$ assumes that every particle is characterized only by its main tune. The spectral weight difference between $\rho_x$ and $\rho_{1x}$ at $\nu - Q_s$ measures the satellites contribution to the $A\bar{x}$ damping mechanism. $\rho_x$ and $\rho_{1x}$ for the 0.5% largest changing energy particles are shown in Fig. 4. Besides the peak at $\nu - Q_s$, in both $\rho_x$ and $\rho_{1x}$ a broad spectral feature at smaller frequency, unfavorable to the $A\bar{x}$ damping mechanism, is observed.

The other contribution of the $J_d$ oscillations to the damping is via the $BJ_d\bar{x}$ term. The resonance condition is $Q_x + Q_t \approx \nu - Q_s$. We define $h(Q)$ as the tune density obtained by shifting each particle's horizontal tune by $Q_t$,

$$\begin{aligned} h(Q) &= \textstyle\sum_i h_i(Q) = \sum_i \int \rho_{J_d i}(Q')\rho_{xi}(Q - Q')dQ' \\ &\approx \textstyle\sum_i \rho_{xi}(Q - Q_{ti}). \end{aligned} \tag{4}$$

$\rho_{J di}(Q) = |\tilde{J}_{di}(Q)|^2$ is the particle's $i$ $J_d$ Fourier spectrum. $h_1(Q)$ is defined by replacing $\rho_{xi}$ with $\rho_{1xi}$ in Eq. 4. As shown in Fig. 4, both $h(Q)$ and $h_1(Q)$ are strongly peaked at the coherent frequency $\nu - Q_s$ and do not display the broad spectral feature seen in $\rho_x(Q)$ and $\rho_{1x}(Q)$ below $\nu - Q_s$. The particles with the tune forming the broad spectral feature of $\rho_{1x}(Q)$ have the main tune $\bar{Q}_x \approx \nu - Q_s - Q_t$, *i.e.*, the tune required for resonance with the $BJ_d\bar{x}$ coupling.

## CONCLUSIONS

Using Synergia with the DMD method the properties of the first SC mode are calculated for a Gaussian bunch propagating through an OFORODO lattice. The off-resonance and the CR cases are compared. While the SC mode's tune and shape are nearly the same, the LD is approximately a factor of 2 larger at CR in the strong SC regime. In the off-resonance case the damping mechanism can be understood within the conventional paradigm. The damping is caused by the resonant energy exchange between the mode and the particles with an incoherent tune spectra equal to the mode's tune shifted by $Q_s$. At CR a large number of particles are trapped around the stable points. Their transverse actions are oscillating with a particle dependent trapping frequency $Q_t$. The spectral properties of the trapped particles with large energy exchange reveal that their tune is additionally shifted from the mode's coherent tune by $Q_t$. This supports an unconventional LD mechanism in which the mode-particle coupling is modulated by the oscillation of the particles' amplitudes.

## ACKNOWLEDGMENT

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