BI-COMPLEX TOOLBOX APPLIED TO GYROMAGNETIC BEAM BREAK-UP*

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Abstract

Transverse instability of a multi-bunch beam in the presence of a longitudinal magnetostatic field and hybrid dipole modes is considered analytically within a single-section model. It incorporates resonant interaction with beam harmonics and eigenmodes, degenerated waves of different polarizations, and the Lorentz' RF force contribution. The analysis is performed in a very compact form using a bi-complex *i,j*-space including four-component collective frequency $\tilde{\nu}$ of the instability. Rotating polarization of the collective field is determined by $\mathrm{Im}_i \mathrm{Im}_j \tilde{\nu}$ that is in agreement with experimental data. The other three components represent detuning of the collective frequency $\mathrm{Re}_i \mathrm{Re}_j \tilde{\nu}$, the "left-hand", and "right-hand" increments $\mathrm{Im}_i \mathrm{Re}_j \tilde{\nu} \pm \mathrm{Im}_j \mathrm{Re}_i \tilde{\nu}$ of the gyro-magnetic BBU effect. The scalar hyper-complex toolbox can be applied to designing of non-ferrite non-reciprocal devices, spin transport, and for characterization of complex transverse dynamics in gyro-devices such as Gyro-TWTs.

INTRODUCTION

Only a few BBU studies give detailed analysis of the instability in the presence of a longitudinal magnetic field [1,2,3]. Some experimental features of the stimulated instability in the presence of a solenoidal field [4] are not completely covered by conventional analytical models [1,3,5]. In particular, the model presented here may be helpful to explain the periodical bouncing and positive gain of stimulated dipole mode instability observed at different magnetic fields above and below the instability threshold [4].

The problem can be conveniently formulated in a 4D hypercomplex space. Conventional quaternions have been applied in physics very productively starting from the prominent Maxwell article [6]. Although the vector quaternions are used commonly, the scalar version of the quaternions is applied here to reduce the problem to a very simple scalar algebraic equation.

ANALYTICAL APPROACH

Self-consistent fields induced in a closed structure can be expanded in series of eigenmodes with variable amplitudes [7]. We apply here the approach describing long-range wake fields induced by a modulated beam in a cavities [8] and slow-wave guides in time-space domain [9,10].

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We treat the instability as being dominated by some complex frequency $\tilde{\omega}$ that has to be defined from a dispersion equation governing the collective process. We consider the BBU effect in the presence of two different polarizations of the dipole mode assuming conventional paraxial approximation $(p_r a)^2 << 1$ (where a is the aperture radius, and p_r is the transverse wavenumber of the r-th mode). We use "fast" and "slow" time variables τ and t respectively, corresponding to the bunch passage through the structure and the field amplitude evolution during the pulse.

Assuming $\tilde{\rho}$ and $\tilde{C}_r$ are the bi-complex [11] variables, the beam transverse motion in presence of focusing is described as follows:

$$\frac{1}{\gamma}\frac{d}{d\tau}\gamma\beta_z\frac{d}{d\tau}\tilde{\rho} + j\Omega\frac{d}{d\tau}\tilde{\rho} - \Omega_\perp|\tilde{\rho}|_j(1+j) = -i\frac{q_o}{2m_o\gamma}\sum_{r,\pm} D_{r,\pm}A_r\tilde{C}_r\exp\left(i(\theta_{r,\pm}\frac{\tau}{\tau_o} - \omega t)\right), \quad (1)$$

where $\tilde{\rho} = \rho_x + j\rho_y$, $\tilde{C}_r = C_r^{(x)} + jC_r^{(y)}$ are the bi-complex displacement and field slowly varying amplitudes in the two orthogonal transverse planes, $x = \mathrm{Re}_i \mathrm{Re}_j \tilde{\rho}$, $y = \mathrm{Re}_i \mathrm{Im}_j \tilde{\rho}$, $\Omega_\perp = \frac{q_o\beta_z c}{2\gamma}\frac{\partial B_z}{\partial z}$, $t = t_{ol} + \tau$ is the "slow" time for the l-th bunch, $\Omega = \frac{q_o B_z}{m_o\gamma}$ is the Larmor frequency, $\theta_{r,\pm} = kL\left(\frac{1}{\pm\beta_r} - \frac{1}{\beta}\right)$ are the phase slippages for forward and backward waves, $\tau_0 = L/\beta c$, and L is the section length, $D_{r,\pm} = \frac{p_r\gamma_r^2}{2h_r}[1 \mp \beta_z\beta_r \pm \Xi_r(\beta_z \mp \beta_r)]$, $\beta_z = v_z/c$, $\beta_r = k_r/h_r$, $\gamma_r^2 = 1/(1-\beta_r^2)$, $p_r = k_r/\beta_r\gamma_r$, $k_r = \omega_r'/c$, $\omega_r = \omega_r' - i\omega_r'' = \omega_r'(1 - i/2Q_r)$, Q_r is the Q-factor, A_r and Ξ_r are the amplitude and hybridization coupling coefficients.

Evolution of the slowly varying field amplitudes $\tilde{C}_r$ can be described with the Vainshtein theory [7] for a resonating structure loaded with the current $\vec{j}(\vec{r},t)$ in the presence of stimulating signal at the frequency ω. For two degenerated eigenmodes polarized in vertical and horizontal planes we can describe the amplitude evolution as follows:

$$\frac{d\tilde{C}_r}{dt} - i(\omega - \omega_r)\tilde{C}_r = -\frac{1}{N_r}\int_V dV\, \vec{j}_\omega \tilde{\vec{E}}^0_{-r}, \quad (2)$$

where $\tilde{\vec{E}}_r = \vec{E}_r^{(x)} + j\vec{E}_r^{(y)} \equiv \left(\vec{E}_r^{0(x)} + j\vec{E}_r^{0(y)}\right)\exp(ih_r z)$, $N_r^{(x,y)} = \varepsilon_0\int dV\vec{E}_r^{(x,y)2}$ is the normalization factor.

At $\Delta\gamma << \gamma$ and $\Omega_\perp << \Omega$ equations (1) and (2) lead to the following system of homogeneous ODEs:

$$\left(\frac{d}{dt}+i(\omega_r-\omega)\right)\tilde{C}_r = I\frac{A_r p_r}{4N_r}\frac{q_o\tau_o^2 L}{4m_o\gamma}\sum_{n=-\infty}^{+\infty}T_n\sum_{r',\pm}A_{r'}D_{r'\pm}\times \tag{3}$$
$$\times\left[\Phi^{(1)\pm}_{rr'n}\tilde{C}_{r'}\exp(-in\omega_o t)+\Phi^{(2)\pm}_{rr'n}\tilde{C}^{*i}_{r'}\exp\left(i(2\omega'-n\omega_o)t\right)\right],$$

where $\tilde{\Phi}^{(1)\pm}_{rr'n}=\int_0^1 d\chi\,\tilde{\varphi}_{r,\pm}(\chi)\exp\left(i(nh_o\pm h_r-h_{r'})\chi L\right)$, and

$$\tilde{\Phi}^{(2)\pm}_{rr'n}=Si\left(T_o(2\omega'-n\omega_o)\right)\int_0^1 d\chi\,\tilde{\varphi}^{*i}_{r,\pm}(\chi)\exp\left(i(nh_o\mp h_r-h_{r'})\chi L\right),$$

$Si(x)=\sin(x)/x$, and $\Omega_o=\Omega\tau_0$.

For BBU interaction dominated by a single mode and beam harmonic we can assume $\tilde{C}_r(t)=\tilde{C}_{init}\exp\left(-i\omega''_r\tilde{\nu}t\right)$. Using algebra [12] and equation (3) one can get the following normalized bi-complex collective frequency $\tilde{\nu}=\tilde{\varpi}/\omega''_r$:

$$\tilde{\nu}=i\left(\frac{I}{G_r}\tilde{\Phi}^{(1)+}_{rr0}-1\right)+a_r, \tag{4}$$

where $a_r=(\omega'_r-\omega)/\omega''_r$ is the generalized frequency detuning,

$$G_r=\omega''_r\left(\frac{A_r^2 p_r}{4N_r}\frac{q_o\tau_o^2 L}{4m_o\gamma}T_0 D_{r+}\right)^{-1}=\frac{m_o\gamma c^2}{q_o R_{\perp r}L/\lambda_r}\cdot\frac{4}{\pi}\cdot\frac{\beta_z^2/\beta_r}{1-\beta_z\beta_r+\Xi_r(\beta_z-\beta_r)},$$

$$R_{\perp r}=\frac{\left(L(\partial E^0_{zr}/\partial y)/k_r\right)^2}{2\omega''_r W_r}=2Lr_\perp$$, where $r_\perp$ is the cavity transverse impedance per unit length [13], and

$$\tilde{\Phi}^{(1)\pm}_{rr0}=\frac{1}{\theta_{r,\pm}-ij\Omega_o}\left[\frac{1-\exp(-i\theta_{r,\pm})}{-i\theta_{r,\pm}}\left(\frac{1}{i\theta_{r,\pm}}+\frac{1}{j\Omega_o}\right)+\frac{1}{i\theta_{r,\pm}}+\frac{1-\exp\left(-i\theta_{r,\pm}-j\Omega_o\right)}{j\Omega_o(i\theta_{r,\pm}+j\Omega_o)}\right].$$

Without the magnetic field one can get the following:

$$\tilde{\Phi}^{(1)+}_{rr0}\xrightarrow[\Omega\to 0]{}\Phi_{1r}(\theta_r)=\left(Si^2(\theta_r/2)-Si(\theta_r)\right)\theta_r^{-1}-2i\left(Si(\theta_r)-\cos^2(\theta_r/2)\right)\theta_r^{-2},$$

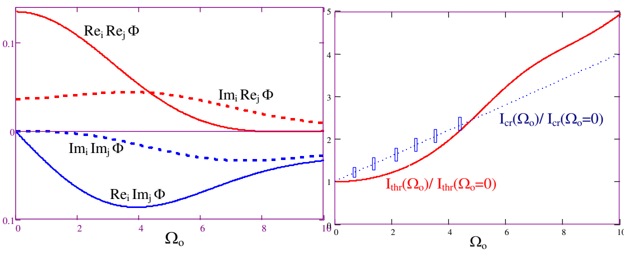
where Φ_{1r} coincides with the conventional single-mode interaction coefficient [13,8] which is maximum at the $\theta_r=2.61$ slippage.

Four components of the interaction coefficient $\tilde{\Phi}^{(1)+}_{rr0}$ are plotted in Fig. 1a.

Using the Euler's formula analog [12] for the commutative hypercomplex numbers the bi-complex amplitude can be represented as follows:

$$\tilde{C}_r=\tilde{C}_{init}\exp\left[-i\omega''_r\left(\mathrm{Re}_i\,\mathrm{Re}_j\,\tilde{\nu}+i\,\mathrm{Im}_i\,\mathrm{Re}_j\,\tilde{\nu}+j\,\mathrm{Re}_i\,\mathrm{Im}_j\,\tilde{\nu}+ij\,\mathrm{Im}_i\,\mathrm{Im}_j\,\tilde{\nu}\right)t\right] \tag{5}$$
$$=\tilde{C}_{init}\exp\left(i\left(\mathrm{Im}_i\,\mathrm{Re}_j\,\tilde{\Phi}^{(1)+}_{rr0}\frac{I}{G_r}-a_r\right)t_1\right)\cdot\exp\left(j\,\mathrm{Re}_i\,\mathrm{Im}_j\,\tilde{\Phi}^{(1)+}_{rr0}\frac{I}{G_r}t_1\right)\times$$
$$\left[\cosh\left(\mathrm{Im}_i\,\mathrm{Im}_j\,\tilde{\Phi}^{(1)+}_{rr0}\frac{I}{G_r}t_1\right)+ij\sinh\left(\mathrm{Im}_i\,\mathrm{Im}_j\,\tilde{\Phi}^{(1)+}_{rr0}\frac{I}{G_r}t_1\right)\right]\exp\left(\left(\mathrm{Re}_i\,\mathrm{Re}_j\,\tilde{\Phi}^{(1)+}_{rr0}\frac{I}{G_r}-1\right)t_1\right),$$

where $t_1=\omega''_r t$ is the dimensionless time.



a) b)

Figure 1: Four components of the bi-complex coefficient $\tilde{\Phi}^{(1)+}_{rr0}$ (a) and the relative increase of the threshold current Eqn. (6) (solid curve) at $\theta_r=2.61$ and critical current [29] (dotted curve) as a function of normalized cyclotron frequency.

Thus each of the four components of the collective frequency has a sound physical meaning: $\mathrm{Im}_i\,\mathrm{Re}_j\,\tilde{\varpi}+\left|\mathrm{Re}_i\,\mathrm{Im}_j\,\tilde{\varpi}\right|=\left[\left(\mathrm{Re}_i\,\mathrm{Re}_j\,\tilde{\Phi}^{(1)+}_{rr0}+\left|\mathrm{Im}_i\,\mathrm{Im}_j\,\tilde{\Phi}^{(1)+}_{rr0}\right|\right)\frac{I}{G_r}-1\right]\omega''_r$ is the maximum increment for two waves with the $\mathrm{Im}_i\,\mathrm{Re}_j\,\tilde{\varpi}\pm\mathrm{Re}_i\,\mathrm{Im}_j\,\tilde{\varpi}$ increments; $\mathrm{Re}_i\,\mathrm{Re}_j\,\tilde{\varpi}=\left(\mathrm{Im}_i\,\mathrm{Re}_j\,\tilde{\Phi}^{(1)+}_{rr0}\frac{I}{G_r}-a_r\right)\omega''_r$ is the collective frequency detuning; and $\mathrm{Im}_i\,\mathrm{Im}_j\,\tilde{\varpi}=\mathrm{Re}_i\,\mathrm{Im}_j\,\tilde{\Phi}^{(1)+}_{rr0}\frac{I}{G_r}\omega''_r$ is the angular velocity of the collective field rotation.

Threshold current in the single-mode approximation can be evaluated as the following:

$$I_{thr}\approx\frac{G_r}{\mathrm{Re}_i\,\mathrm{Re}_j\,\tilde{\Phi}^{(1)+}_{rr0}+\left|\mathrm{Im}_i\,\mathrm{Im}_j\,\tilde{\Phi}^{(1)+}_{rr0}\right|}. \tag{6}$$

The hyperbolic functions in Eqn. (5) with $IG_r^{-1}\,\mathrm{Im}_i\,\mathrm{Im}_j\,\tilde{\Phi}^{(1)+}_{rr0}t_1$ argument indicate $\mathrm{Im}_i\,\mathrm{Re}_j\,\tilde{\varpi}\pm\mathrm{Re}_i\,\mathrm{Im}_j\,\tilde{\varpi}$ increment split. The presence of two waves is due to a combinational effect of coupling between the collective rotation and instability growth. A similar two-frequency behaviour demonstrates, for example, spontaneous radiation of an electron beam in a twisted undulator [14]. For practical estimations of the threshold Eqn. (6) at moderate magnetic fields with $\Omega_o<4$ and to avoid uncertainties in slippage and the increments it is convenient to impose an additional condition: $\mathrm{Im}_i\,\mathrm{Im}_j\,\tilde{\Phi}^{(1)+}_{rr0}=0$. Fig. 1b shows the threshold current plot calculated with Eqn. (6) versus cyclotron frequency and also the same plot for critical current reproduced from experiment [4]. The discrepancy between these two curves is caused by single-mode approximation as well as the difference in the dynamics near the threshold and at critical condition of beam blow-up when beam interception occurs (see below).

In the experiment [4] a circular disk-loaded waveguide supplied with dipole mode couplers on the input and output ends was inserted into a solenoid. The linac driven section has been stimulated by external RF signal at a frequency that is close to the dipole mode frequency. At substantial magnetic fields (0.04T-0.13T) and beam currents (fraction of the critical current) the transmission of the signal waveform has been distorted and repetitively delayed by (2-20) μs depending on the beam current, the signal frequency, and the magnetic field.

Indeed, according to Eqn. (7) the angular velocity of the rotating collective field is proportional to the current and, at $\Omega_o \leq 3$, is approximately proportional to the magnetic field (see Fig 1a.). For example, at $I \approx I_{thr}$ the polarization angle of the collective field can be evaluated as follows:

$$\varphi(t) \approx \frac{\mathrm{Re}_i\,\mathrm{Im}_j\,\tilde{\Phi}^{(1)+}_{rr0}}{\mathrm{Re}_i\,\mathrm{Re}_j\,\tilde{\Phi}^{(1)+}_{rr0} + \left|\mathrm{Im}_i\,\mathrm{Im}_j\,\tilde{\Phi}^{(1)+}_{rr0}\right|}\omega''_r t + \varphi_0 \qquad (7)$$

That means that for the S-band linac with f_r=4.2 GHz dipole mode frequency and Q_r=15000 Q-factor, the polarization rotates by ~180° every ≈2.76 μs at θ_r=2.61 and Ω_o=1.5 and correspondingly induces periodical signal in the output port. The signal distortion is caused by polarization rotation of the collective field speckling the probe having fixed orientation of the optimum coupling.

Using model of discrete bunches one can apply the solution above to derive the transverse displacement of the n-th bunch:

$$\tilde{\rho}_n(\chi) = (\tilde{b} + \tilde{d}\chi)\exp(-i\omega nT_b) + \frac{I}{G_r(1+ia_r)}\tilde{\mathrm{P}}(\theta_r,\chi) \times$$
$$\left[\frac{\left(\tilde{\Phi}^{(1)+}_{rr0} I\left(G_r(1+ia_r)\right)^{-1}\right)^n - 1}{\tilde{\Phi}^{(1)+}_{rr0} I\left(G_r(1+ia_r)\right)^{-1} - 1}\exp\left(-i\omega nT_b\right) - \exp\left(\omega''_r nT_b\left(\frac{\tilde{\Phi}^{(1)+}_{rr0} I}{G_r} - 1\right) - i\omega'_r nT_b\right)\right], \qquad (8)$$

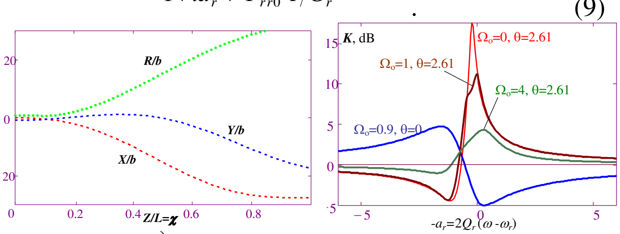
where

$$\tilde{\mathrm{P}}(\theta,\chi) = \tilde{\varphi}_{r,+}(\chi)\exp(-i\theta/2)\left[\tilde{b}\,Si(\theta/2) - i\tilde{d}\left(Si(\theta/2) - \exp(-i\theta/2)\right)/\theta\right],$$

$\tilde{\rho}_n(0) = \tilde{b}\exp(-i\omega nT_b)$, $d\tilde{\rho}_n/d\chi(0) = \tilde{d}\exp(-i\omega nT_b)$ are the initial conditions for the model with external stimulation at frequency ω.

Below threshold the signal amplification gain $K[dB] = 20\lg\|\tilde{K}\|$ plotted in Fig 2b is evaluated as magnitude of trajectories evolving in time at $I \leq I_{thr1}$ and $n >> 1$ is evaluated as follows:

$$\tilde{K} \approx 1 + \frac{\exp(-i\theta_r/2)Si(\theta_r/2)\tilde{\Phi}^{(1)+}_{rr0}\,I/G_r}{1 + ia_r + \tilde{\Phi}^{(1)+}_{rr0}\,I/G_r}. \qquad (9)$$



a) b)

Figure 2: Normalized trajectories at beam blow-up (a) f_b =2.8 GHz, I/G_r=34, Ω_o=4.2, a_r=3, f_r=4.2GHz, Q_r =15104, t =2 μs. and gain below threshold (b) versus detuning at I/G_r =6.7 and different magnetic fields.

The gain curves explain both substantial gain and polarization dependency vs. detuning of the transmitted signal below the threshold observed in [4].

DISCUSSION

Exactly the same result for the instability frequency and increments we obtained as a benchmark using conventional vectors and matrixes. However, it would be not possible to get similar to (8) and (9) result in a closed compact form using conventional approaches, i.e. complex matrices or non-commutative quaternions. The paper results can be treated also as a benchmark for the algebra of scalar quaternions applied to modeling of coupled transverse beam dynamics in time-varying fields and compared to the conventional non-commutative algebra of vectors and matrices. Potential convenience and effectiveness of the bi-complex variables are related to physical transparency and exceptional compactness. It is accomplished due to encapsulation of the algebra of spin matrices into the commutative i, j-algebra [12]. One can simplify routine operations related to the hypercomplex scalars by implementing corresponding rules as a symbolic application.

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