SECTOR MAGNETS OR TRANSVERSE ELECTROMAGNETIC FIELDS IN CYLINDRICAL COORDINATES *

T. Zolkin†, Fermilab, Batavia IL 60510, USA

Abstract

Laplace's equation in normalized cylindrical coordinates is considered for scalar and vector potentials describing electric or magnetic fields with invariance along the azimuthal coordinate [1]. A series of special functions are found which when expanded to lowest order in power series in radial and vertical coordinates ($\rho = 1$ and $y = 0$) replicate harmonic homogeneous polynomials in two variables. These functions are based on radial harmonics found by Edwin M. McMillan forty years ago. In addition to McMillan's harmonics, a second family of radial harmonics is introduced to provide a symmetric description between electric and magnetic fields and to describe fields and potentials in terms of the same functions. Formulas are provided which relate any transverse fields specified by the coefficients in the power series expansion in radial or vertical planes in cylindrical coordinates with the set of new functions.

This result is important for potential theory and for theoretical study, design and proper modeling of sector dipoles, combined function dipoles and any general sector element for accelerator physics and spectrometry.

INTRODUCTION

The description of sector magnets, any curved magnet symmetric along its azimuthal (longitudinal) cylindrical coordinate, is an important issue. Every modern accelerator code includes such elements, the most important being combined function dipoles. A widely used method, which goes back to Karl Brown's 1968 paper [2], is based on a solution of Laplace's equation for a scalar potential using a power series in cylindrical coordinates. A similar approach applied to Laplace's equation for the longitudinal component of a vector potential can be found for example in [3]. The same approach appears in more recent books, e.g. [4].

Two major bottlenecks should be noticed. First, if one looks for a solution in the form of a series, then these series must be truncated. In our case truncation means that potentials no longer satisfy Laplace's equation. Of course potentials can "satisfy" Laplace's equation up to any desired order by keeping more and more terms in the expansion, but they are not exact. More importantly, the recurrence equation is undetermined: in every new order of recurrence one has to assign an arbitrary constant, which will affect all other higher order terms. This ambiguity leads to the fact that there is no preferred, unique choice of basis functions; it makes it difficult to compare accelerator codes, since different assumptions might be used for representations of basis functions.

This indeterminacy has a simple geometrical illustration. Looking for a field with e.g. pure normal dipole component on a circular designed equilibrium orbit in lowest order, one can come up with an almost arbitrary shape of the magnet's north pole if its south pole is symmetric with respect to the midplane. In the case of a dipole, the series can be truncated by keeping only its dipole component. For higher order multipoles in cylindrical coordinates truncation without violation of Laplace's equation is not possible.

While working on an implementation of sector magnets for Synergia, I found assumptions which let me sum series for pure electric and magnetic skew and normal multipoles. Looking further for symmetry in the description allowed me to generate a family of solutions in which all the series could be summed, so that no truncation was required. While discussing my results with Sergei Nagaitsev, he brought my attention to an article by McMillan written in 1975 [5]. As I found later, the same result was independently obtained by S. Mane and published in the same journal about 20 years later [6] without citing McMillan's original work. It made me to write this article in order to bring attention back to thees forgotten results.

Joining my results to McMillan's, I would like to present a new representation for multipole expansions in cylindrical coordinates. Any transverse field can be expanded in terms of these functions and related to power series expansions in horizontal or vertical planes. The new approach does not contradict previous results but embraces them. The ambiguity in choice of coefficients and the problem of truncation are resolved. Thus it can be employed for theoretical studies, design and simulation of sector magnets.

The expansion of static electromagnetic fields with rotational symmetry about vertical axis, y, is derived in right-handed normalized cylindrical coordinates $(\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_\theta)$. Sometimes sector coordinates $(\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_\theta)$ are used instead. They defined as another orthogonal right-handed normalized system of coordinates with $\rho = x + 1$. It is in sector coordinates scalar potential Φ and only one nonvanishing component of vector potential A_θ, when expanded at $x, y = 0$, reproduce harmonic homogeneous polynomials in the lowest order of expansion.

The paper is structured as follows. In first section we consider pure transverse electric or magnetic fields in cylindrical coordinates. In the second one we compare new results with traditional approach of power series ansatz. Tables 1–2 are supplementary materials with sector harmonics and its relationship with power series expansion of fields.

* Fermi National Accelerator Laboratory (Fermilab) is operated by Fermi Research Alliance, LLC. for the U.S. Department of Energy under contract DE-AC02-07CH11359.

† zolkin@fnal.gov

MULTIPOLES IN CYLINDRICAL COORDINATES

In the normalized right-handed cylindrical coordinate system, when cylindrical symmetry $\partial/\partial\theta = 0$ is imposed, the Laplace equations reduce to

$$\begin{aligned}\triangle\Phi &\equiv \nabla\cdot\nabla\Phi = \triangle_\perp\Phi + \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho}\\ &= \frac{\partial^2\Phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{\partial^2\Phi}{\partial y^2} = 0,\\ \triangle\mathbf{A} &\equiv \nabla(\nabla\cdot\mathbf{A}) - \nabla\times(\nabla\times\mathbf{A}) = \left(\triangle A_\theta - \frac{A_\theta}{\rho^2}\right)\hat{\mathbf{e}}_\theta\\ &= \left(\frac{\partial^2 A_\theta}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial A_\theta}{\partial\rho} + \frac{\partial^2 A_\theta}{\partial y^2} - \frac{A_\theta}{\rho^2}\right)\hat{\mathbf{e}}_\theta = 0\,.\end{aligned}$$

Compared to the case with Cartesian coordinates these equations look quite different from each other. In order to retain the symmetry one can note that

$$(\triangle\mathbf{A})_\theta = \frac{1}{\rho}\left[\frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial y^2}\right](\rho A_\theta)\,.$$

Thus looking for the solution in a form similar to harmonic homogeneous polynomials

$$\begin{aligned}\Phi &= -\sum_{k=0}^{n}\frac{\mathcal{F}_{n-k}(\rho)}{(n-k)!}\frac{y^k}{k!}\left(\mathrm{C}_n\,\sin\frac{k\pi}{2} + \mathrm{D}_n\,\cos\frac{k\pi}{2}\right),\\ A_\theta &= -\sum_{k=0}^{n}\frac{1}{\rho}\frac{\mathcal{G}_{n-k}(\rho)}{(n-k)!}\frac{y^k}{k!}\left(\mathrm{C}_n\,\cos\frac{k\pi}{2} - \mathrm{D}_n\,\sin\frac{k\pi}{2}\right),\end{aligned}$$

where C_n and D_n are normal and skew multipole coefficients of the expansion, one can find that functions $\mathcal{F}_n(\rho)$ and $\mathcal{G}_n(\rho)$ are related to each other through

$$\mathcal{G}_{n-1} = \frac{1}{n}\,\rho\,\frac{\partial\mathcal{F}_n}{\partial\rho} \qquad\qquad \mathcal{F}_{n-1} = \frac{1}{n}\frac{1}{\rho}\frac{\partial\mathcal{G}_n}{\partial\rho}$$

and satisfying recurrence equations

$$\begin{aligned}\frac{\partial^2\mathcal{F}_n(\rho)}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\mathcal{F}_n(\rho)}{\partial\rho} &= n\,(n-1)\,\mathcal{F}_{n-2}(\rho),\\ \frac{\partial^2\mathcal{G}_n(\rho)}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial\mathcal{G}_n(\rho)}{\partial\rho} &= n\,(n-1)\,\mathcal{G}_{n-2}(\rho)\,.\end{aligned}$$

In addition, $\mathcal{F}_n(\rho)$ and $\mathcal{G}_n(\rho)$ allow us to construct lowering operators

$$\begin{aligned}\mathcal{F}_n &= \frac{1}{(n+1)(n+2)}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial}{\partial\rho}\right)\right]\mathcal{F}_{n+2}\,,\\ \mathcal{G}_n &= \frac{1}{(n+1)(n+2)}\left[\rho\,\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\right]\mathcal{G}_{n+2}\,,\end{aligned}$$

and corresponding raising operators where the lower limits take care of two arbitrary constants of integration

$$\begin{aligned}\mathcal{F}_n &= n\,(n-1)\int_1^\rho\frac{1}{\rho}\int_1^\rho \rho\,\mathcal{F}_{n-2}\,\mathrm{d}\rho\,\mathrm{d}\rho\,,\\ \mathcal{G}_n &= n\,(n-1)\int_1^\rho \rho\int_1^\rho\frac{1}{\rho}\,\mathcal{G}_{n-2}\,\mathrm{d}\rho\,\mathrm{d}\rho\,.\end{aligned}$$

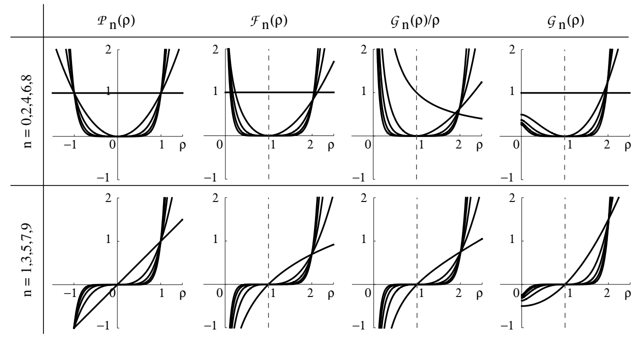

Figure 1: First five even (top row) and odd (bottom row) members of power functions $\mathcal{P}_n(\rho) = \rho^n$, $\mathcal{F}_n(\rho)$, $\frac{\mathcal{G}_n(\rho)}{\rho}$ and $\mathcal{G}_n(\rho)$ from the left to the right respectively.

These operators can be used to recursively calculate all members of $\mathcal{F}-$ and $\mathcal{G}-$functions. An additional constraint to terminate recurrences defines lowest orders as

$$\mathcal{F}_0 = 1,\quad \mathcal{F}_1 = \ln\,\rho,\quad \mathcal{G}_0 = 1,\quad \mathcal{G}_1 = (\rho^2-1)/2.$$

The first ten members of $\mathcal{F}_n$ and $\mathcal{G}_n$ are shown in Fig. 1. Note that Taylor series of these functions at $\rho = 1$ are $\mathrm{T}(\mathcal{F}_n, \mathcal{G}_n) = x^n + O(x^{n+1})$. The difference relation for $\mathcal{F}_n$ and first members have been found by E.M. McMillan and I would like to acknowledge it by given them a name of *McMillan radial harmonics*. In addition to his results we introduced *adjoint McMillan radial harmonics*, $\mathcal{G}_n$, in order to provide the symmetry in description between electric and magnetic fields. Finally, in order to define the set of functions for pure sector multipoles (see Fig. 2) we will define *sector harmonics*:

$$\begin{aligned}\mathcal{A}^{(\mathrm{e})}_n(\rho,y) &= \sum_{k=0}^{n}\binom{n}{k}\cos\frac{k\pi}{2}\,\mathcal{F}_{n-k}(\rho)\,y^k,\\ \mathcal{A}^{(\mathrm{m})}_n(\rho,y) &= \sum_{k=0}^{n}\binom{n}{k}\cos\frac{k\pi}{2}\,\frac{\mathcal{G}_{n-k}(\rho)}{\rho}\,y^k,\\ \mathcal{B}^{(\mathrm{e})}_n(\rho,y) &= \sum_{k=0}^{n}\binom{n}{k}\sin\frac{k\pi}{2}\,\mathcal{F}_{n-k}(\rho)\,y^k,\\ \mathcal{B}^{(\mathrm{m})}_n(\rho,y) &= \sum_{k=0}^{n}\binom{n}{k}\sin\frac{k\pi}{2}\,\frac{\mathcal{G}_{n-k}(\rho)}{\rho}\,y^k,\end{aligned}$$

obeying differential relations

$$\begin{aligned}n\,(\mathcal{A},\mathcal{B})^{(\mathrm{e})}_{n-1} &= \pm\frac{\partial\,(\mathcal{B},\mathcal{A})^{(\mathrm{e})}_n}{\partial y} = \frac{1}{\rho}\frac{\partial\left[\rho\,(\mathcal{A},\mathcal{B})^{(\mathrm{m})}_n\right]}{\partial\rho}\,,\\ n\,(\mathcal{A},\mathcal{B})^{(\mathrm{m})}_{n-1} &= \pm\frac{1}{\not\rho}\frac{\partial\left[\not\rho\,(\mathcal{B},\mathcal{A})^{(\mathrm{m})}_n\right]}{\partial y} = \frac{\partial\,(\mathcal{A},\mathcal{B})^{(\mathrm{e})}_n}{\partial\rho}\,.\end{aligned}$$

Last pair of equations show that these harmonics describe not only potentials but components of a field as well, since

$$\mathbf{E} = -\frac{\partial\Phi}{\partial\rho}\,\hat{\mathbf{e}}_\rho - \frac{\partial\Phi}{\partial y}\,\hat{\mathbf{e}}_y\,,\qquad \mathbf{B} = \frac{1}{\rho}\frac{\partial\,\rho A_\theta}{\partial y}\,\hat{\mathbf{e}}_\rho - \frac{1}{\rho}\frac{\partial\,\rho A_\theta}{\partial\rho}\,\hat{\mathbf{e}}_y\,.$$

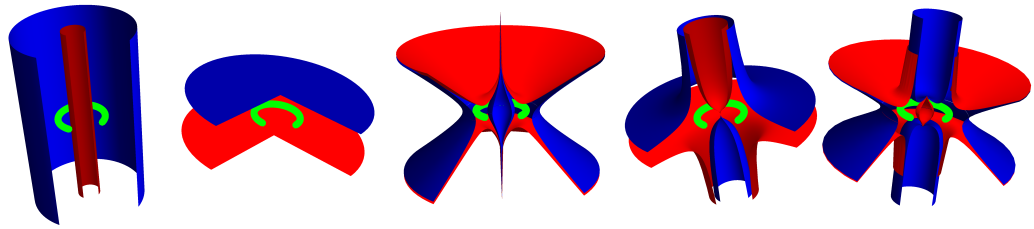

Figure 2: 3D models of pure 2n-pole sector magnets. North (red) and south (blue) poles are given by constant levels of $(\mathcal{B},\mathcal{A})_n^{(e)} = \mp\text{const}$ for normal and skew magnets respectively. From the left to the right: skew S-dipole, normal S-dipole, skew S-quadrupole, normal S-quadrupole and skew S-sextupole. The circular design orbit $\rho = 1$ is shown in green color.

RECURRENCE EQUATIONS IN SECTOR COORDINATES

An alternative approach to finding expansions for potentials is to use a general power series ansatz

$$\Phi = -\sum_{m,n\geq 0}^{\infty} V_{m,n}\frac{x^m}{m!}\frac{y^n}{n!},$$
$$A_\theta = -\sum_{m,n\geq 0}^{\infty} \frac{1}{1+x} V_{m,n}\frac{x^m}{m!}\frac{y^n}{n!}.$$

In Cartesian coordinates the substitution gives the recurrence relation $V_{m+2,n} + V_{m,n+2} = 0$. It immediately defines all coefficients, and up to a common factor, they coincide with the coefficients of harmonic homogeneous polynomials. In sector coordinates the same substitution gives two new undetermined recurrences, and, for scalar and vector potentials they are respectively

$$V_{m+2,n} + V_{m,n+2} = -(m \pm 1)\, V_{m+1.n} - m\, V_{m-1,n+2}.$$

In order to solve these recurrences, one can look for a solution where each term can be expressed in the form

$$V_{i,j} = V^*_{i,j} + V^{(i+j-1)}_{i,j} + V^{(i+j-2)}_{i,j} + V^{(i+j-3)}_{i,j} + \dots$$

where starred variables are the "design" terms given by pure multipole fields satisfying $V^*_{m+2,n} + V^*_{m,n+2} \equiv 0$. Other coefficients $V^{(k)}_{i,j}$ are terms induced by lower k-th order pure multipoles due to recurrence, and for a particular $2n$-pole for $i + j > n$ are subject to be determined.

In order to use these equations one will have to truncate a recurrence. As a result the potentials representing magnets no longer satisfy Laplace's equation. This violates the "physics" and should be avoided. While potentials can be approximated with any precision by keeping an appropriate number of terms, when solving the recurrence, at each new order one will find that an arbitrary constant $\alpha_i \in (0;1)$ should be introduced since the system is undetermined. An additional assumption, $(A_\theta, \Phi)\,|_{x=0} \propto y^n$ allows us to truncate or sum the series. The resulting solutions would then coincide with sector harmonics obtained above.

CONCLUSION

The scalar and vector Laplace's equations for static transverse electromagnetic fields in normalized curvilinear orthogonal coordinates with constant curvature $\kappa = 1$ and cylindrical symmetry are solved. The set of solutions named sector harmonics (first few members are listed in Table 1). It should not be confused with cylindrical harmonics where ρ-dependent terms are given by Bessel functions. In contrast, the radial part is given by a set of harmonics, independently introduced by E.M. McMillan in a "forgotten" article, and adjoint radial harmonics described in this work. When expanded around a circular design orbit ($\rho = 1$, $y = 0$), sector harmonics resemble harmonic homogeneous polynomials of two variables which are solutions in Cartesian geometry. This set of functions has two major advantages over the traditional approach, widely used in the accelerator community, of using recurrences based on a power series ansatz. They do not require truncation and satisfy Laplace's equation exactly. In addition they provide a well defined full basis of functions which can be related to any field by expansion in radial or vertical planes, see Table 2. Thus, I would like to suggest the set of sector harmonics as a new basis for the description and design of sector magnets with translational symmetry along the azimuthal coordinate.

ACKNOWLEDGMENT

The author would like to thank Leo Michelotti, Eric Stern and James Amundson for their discussions and valuable input, Alexey Burov for encouraging me to find a full family of solutions, Valeri Lebedev whose solution for electrostatic quadrupole led me to a generalization, just as was the case with E. M. McMillan and F. Krienen, and Sergei Nagaitsev who introduced me to McMillan's article, which helped me with the symmetric description of electromagnetic fields.

REFERENCES

[1] T. Zolkin, arXiv:1603.03451, 2016.
[2] K. L. Brown, *Adv. Part. Phys.*, vol. 1, p. 71, 1968.
[3] E. Forest, *Beam dynamics, Vol. 8*: CRC Press, 1998.
[4] H. Wiedemann, *Particle Accelerator Physics*: Springer, 2015.
[5] E. M. McMillan, *Nucl. Instrum. Meth.*, vol. 127, p. 471, 1975.
[6] S. Mane, *Nucl. Instrum. Meth.*, vol. 321, p. 365, 1992.

Table 1: Sector Harmonics

	n	
$\mathcal{A}_n^{(e)}$	0	1
	1	$\ln\rho$
	2	$\left[\frac{\rho^2-1}{2} - y^2\right] - \ln\rho$
	3	$\left[-3\,\frac{\rho^2-1}{2}\right] + 3\,\left(\frac{\rho^2+1}{2} - y^2\right)\ln\rho$
	4	$\left[\frac{3(\rho^4+4\rho^2-5)}{8} - 6\,\frac{\rho^2-1}{2}\,y^2 + y^4\right] - 3\,\left(\frac{1}{2} + \rho^2 - 2\,y^2\right)\,\ln\rho$
$\mathcal{A}_n^{(m)}$	0	$\frac{1}{\rho}$
	1	$\frac{1}{\rho}\left[\frac{\rho^2-1}{2}\right]$
	2	$\frac{1}{\rho}\left\{\left[-\frac{\rho^2-1}{2} - y^2\right] + \rho^2\ln\rho\right\}$
	3	$\frac{1}{\rho}\left\{\left[\frac{3(\rho^2+1)}{4}\frac{\rho^2-1}{2} - 3\,\frac{\rho^2-1}{2}\,y^2\right] - \frac{3}{2}\,\rho^2\ln\rho\right\}$
	4	$\frac{1}{\rho}\left\{\left[-\frac{3(5\rho^4-4\rho^2-1)}{8} + 6\,\frac{\rho^2-1}{2}\,y^2 + y^4\right] + \frac{3(2+\rho^2-4\,y^2)}{2}\,\rho^2\ln\rho\right\}$
$\mathcal{B}_n^{(e)}$	0	0
	1	y
	2	$y\,[2\,\ln\rho]$
	3	$y\left\{\left[3\,\frac{\rho^2-1}{2} - y^2\right] - 3\,\ln\rho\right\}$
	4	$y\left\{\left[-12\,\frac{\rho^2-1}{2}\right] + 4\,\left(3\,\frac{\rho^2+1}{2} - y^2\right)\ln\rho\right\}$
$\mathcal{B}_n^{(m)}$	0	0
	1	$\frac{y}{\rho}$
	2	$\frac{y}{\rho}\left[2\,\frac{\rho^2-1}{2}\right]$
	3	$\frac{y}{\rho}\left\{\left[-3\,\frac{\rho^2-1}{2} - y^2\right] + 3\,\rho^2\ln\rho\right\}$
	4	$\frac{y}{\rho}\left\{\left[4\,\frac{3(\rho^2+1)}{4}\frac{\rho^2-1}{2} - 4\,\frac{\rho^2-1}{2}\,y^2\right] - 6\,\rho^2\ln\rho\right\}$

Table 2: Relationship between coefficients determining the strength of pure normal and skew sector multipoles and power series expansion of field in radial and vertical planes at $x, y = 0$ ($\rho = 1$).

	C_n		D_n	
n	$x = 0$	$y = 0$	$x = 0$	$y = 0$
1	F_y	F_y	F_x	F_x
2	$\partial_y F_x$	$\partial_x F_y$	$-\partial_y F_y$	$\partial_x F_x + F_x$
3	$-\partial_y^2 F_y$	$\partial_x^2 F_y + \partial_x F_y$	$-\partial_y^2 F_x$	$\partial_x^2 F_x + \partial_x F_x - F_x$
4	$-\partial_y^3 F_x$	$\partial_x^3 F_y + \partial_x^2 F_y - \partial_x F_y$	$\partial_y^3 F_y$	$\partial_x^3 F_x + 2\,\partial_x^2 F_x - \partial_x F_x + F_x$
5	$\partial_y^4 F_y$	$\partial_x^4 F_y + 2\,\partial_x^3 F_y - \partial_x^2 F_y + \partial_x F_y$	$\partial_y^4 F_x$	$\partial_x^4 F_x + 2\,\partial_x^3 F_x - 3\,\partial_x^2 F_x + 3\,\partial_x F_x - 3\,F_x$