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COMPUTING EIGEN-EMITTANCES FROM TRACKING DATA*

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Abstract

In a strongly nonlinear system the particle distribution in the phase space may develop long tails which contribution to the covariance (sigma) matrix should be suppressed for a correct estimate of the beam emittance. A method is offered based on Gaussian approximation of the original particle distribution in the phase space (Klimontovich distribution) which leads to an equation for the sigma matrix which provides efficient suppression of the tails and cannot be obtained by simply introducing weights. This equation is easily solved by iterations in the multi-dimensional case. It is also shown how the eigen-emittances and coupled optics functions can be retrieved from the sigma matrix in a strongly coupled system.

INTRODUCTION

Finding normal mode emittances (eigen-emittances) or just second order moments from experimental or simulations data is needed for many applications, most notably in analysis of particle cooling. In the cooling process the non-Gaussian tails can develop producing a significant contribution to the second moments of the distribution. Of course we can make cuts but since the emittances are not known in advance the procedure is ambiguous.

Furthermore, besides suppression of the halo contribution, an acceptable method must also provide the exact result in absence of the halo. In the next section it is shown how to do this.

In the third section we show – using the theory developed by V. Lebedev & A.Bogacz [1] – how to find eigen-emittances and optics functions from a known covariance matrix in the case of coupled oscillations.

Finally, in the Appendix we make an estimate of the error in eigen-emittances if the mechanical momenta are used instead of the canonical ones.

GAUSSIAN FIT OF THE KLIMONTOVICH DISTRIBUTION

First let us introduce notation conventions: underlined characters will denote (column) phase space vectors, whereas upright capital letters will be used to designate matrices.

Following [2] let us choose the path length s along the reference orbit as the independent variable and dynamical variables in the form:

$$\underline{z} = \{x, P_x, y, P_y, s - c\beta_0 t, \delta\} \tag{1}$$

where $P_{x,y}$ are canonical momenta normalized by the reference value $p_0 = mc\beta_0\gamma_0$:

$$P_x = (p_x + \frac{e}{c}A_x)/p_0 \tag{2}$$

with $p_{x,y}$ and $A_{x,y}$ being the components of the mechanical momentum and magnetic vector potential respectively (we use Gaussian units). Finally,

$$\delta = (\gamma - \gamma_0)/\beta_0^2\gamma_0 \tag{3}$$

Suppose that from measurements or simulations we have a set of N particles positions in the phase space, $\underline{z}^{(k)}$, $k = 1,\ldots, N$, and our task is to find the normal mode emittances. Let us first assume that the given distribution does not contain long tails (outliers) and therefore we can use simple averaging to find elements of the covariance matrix Σ:

$$\Sigma_{i,j} = \frac{1}{N}\sum_{k=1}^{N}\zeta_i^{(k)}\zeta_j^{(k)}, \quad \underline{\zeta}^{(k)} = \underline{z}^{(k)} - \underline{a}, \quad \underline{a} = \frac{1}{N}\sum_{k=1}^{N}\underline{z}^{(k)} \tag{4}$$

Now consider particle distribution in the phase space which is sometimes referred to as the Klimontovich distribution:

$$G(\underline{z}) = \frac{1}{N}\sum_{k=1}^{N}\delta_{6D}(\underline{z} - \underline{z}^{(k)}) \equiv \frac{1}{N}\sum_{k=1}^{N}\prod_{i=1}^{6}\delta(z_i - z_i^{(k)}) \tag{5}$$

where δ_{6D} is the six-dimensional Dirac's δ-function.

Our task is to approximate distribution (5) with a smooth function. We will employ Gaussian distribution, though other functions (e.g. parabolic) can be used:

$$F(\underline{\zeta}) = \frac{\eta}{(2\pi)^{n/2}\sqrt{\det\Sigma}}\exp[-\frac{1}{2}(\underline{\zeta}, \Sigma^{-1}\underline{\zeta})] \tag{6}$$

where n is the dimensionality of the problem ($n = 6$ in our case) and $(\underline{a}, \underline{b})$ means a scalar product. Parameter η can be considered as the fraction of particles in the beam core.

For a moment let us replace point-like particles in distribution G with spheres of radius ρ so that G was integrable with square and later put $\rho \to 0$. Keeping ρ finite, the fitting can be formulated as a minimization problem,

$$\begin{aligned}&\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}|F - G|^2\, dz_1\ldots dz_n = \\ &\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}(F^2 - 2FG)dz_1\ldots dz_n + \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}G^2 dz_1\ldots dz_n \to \min\end{aligned} \tag{7}$$

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Since the last integral does not depend on the fitting parameters the problem can be reformulated as a search for the maximum of the first term taken with the opposite sign. As G enters this term linearly we can put now $\rho \to 0$ and perform integration. This leads to the merit function

$$H(\Sigma, \underline{a}, \eta) = \frac{\eta}{\sqrt{\det \Sigma}} \left\{ \frac{1}{N} \sum_{k=1}^{N} \exp[-\frac{1}{2}(\underline{\zeta}^{(k)}, \Sigma^{-1} \underline{\zeta}^{(k)})] - \frac{\eta}{2^{n/2+1}} \right\} \quad (8)$$

Iterative Procedure

Imposing the requirement that all derivatives of the merit function w.r.t. fitting parameters turn zero we can obtain the following set of equations (see [3] for details):

$$\Sigma_{ij} = \frac{\frac{1}{N} \sum_{k=1}^{N} \zeta_i^{(k)} \zeta_j^{(k)} \exp[-\frac{1}{2}(\underline{\zeta}^{(k)}, \Sigma^{-1} \underline{\zeta}^{(k)})]}{\frac{1}{N} \sum_{k=1}^{N} \exp[-\frac{1}{2}(\underline{\zeta}^{(k)}, \Sigma^{-1} \underline{\zeta}^{(k)})] - \frac{\eta}{2^{n/2+1}}},$$

$$\underline{a} = \frac{\sum_{k=1}^{N} \underline{z}^{(k)} \exp[-\frac{1}{2}(\underline{\zeta}^{(k)}, \Sigma^{-1} \underline{\zeta}^{(k)})]}{\sum_{k=1}^{N} \exp[-\frac{1}{2}(\underline{\zeta}^{(k)}, \Sigma^{-1} \underline{\zeta}^{(k)})]}, \quad (9)$$

$$\eta = \frac{2^{n/2}}{N} \sum_{k=1}^{N} \exp[-\frac{1}{2}(\underline{\zeta}^{(k)}, \Sigma^{-1} \underline{\zeta}^{(k)})].$$

Please note that the first of these equations cannot be obtained by introducing weights in the definition (4).

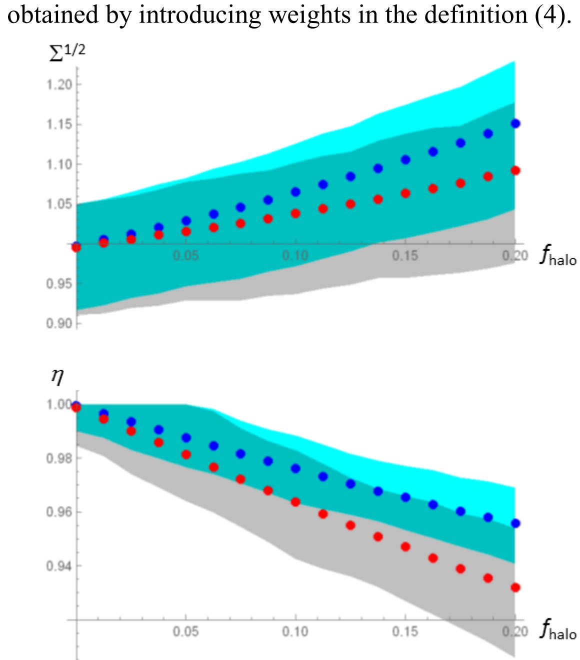

Figure 1: Average over 100 realizations beam size (top) and fraction of particles in the core (bottom) vs. fraction of particles in the halo obtained with $\eta = 1$ fixed during iterations (blue dots) and with fitted η (red dots). The shadowed areas show the spread in these values.

In principle parameter η can be excluded by plugging the last of Eqs. (9) into the first one, but then the control over its value will be lost: due to the statistical nature of the input data it may exceed 1 in the process leading to wrong results. So the loop over this parameter should be the outmost one if any.

The iterations over the Σ-matrix have a tendency to oscillate so the convergence can be accelerated by introducing some damping.

Figure 1 shows the results of simulations with randomly generated 1D distributions of 2000 particles which are a superposition of two normal distributions: one with $\sigma = 1$ (core) and the other with $\sigma = 3$ (halo of relative intensity f_{halo}). Each point presents an average over 100 seeds. Red color shows the results of iterations with variable η while the blue color corresponds to iterations with $\eta = 1$. The shown η values in the latter case are obtained a posteriori using the last of Eqs. (9).

It can be seen that inclusion of η in iterations noticeably improves the halo suppression but introduces a larger spread in the results. In both cases the obtained σ values are much closer to that for the core compared to simple r.m.s. value which for $f_{halo} = 0.2$ is $\Sigma^{1/2} = 1.61$.

EMITTANCES FROM Σ-MATRIX

Known the Σ-matrix we can not only find the normal mode emittances but also obtain information on the beta and dispersion functions. Taking the cue from Ref. [1] let us consider the matrix product

$$\Omega = \Sigma \mathrm{S} \quad (10)$$

with S being the symplectic unity matrix

$$\mathrm{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (11)$$

It can be shown [3] that matrix Ω has purely imaginary conjugate pairs of eigenvalues whose absolute values give the normal mode emittances.

For eigenvectors belonging to the same normal mode m we can put

$$\underline{v}_{2m} = \underline{v}^*_{2m-1} \quad (12)$$

where asterisk denotes complex conjugation and impose the following normalization:

$$(\underline{v}^*_{2m-1}, \mathrm{S}\, \underline{v}_{2m-1}) = -2i \quad (13)$$

Then the coupled optics functions (called Mais-Ripken functions) can be found as [3]

$$\beta_{xm} = |(\underline{v}_{2m-1})_1|^2,\ \beta_{ym} = |(\underline{v}_{2m-1})_3|^2,\ \beta_{sm} = |(\underline{v}_{2m-1})_5|^2 \quad (14)$$

Due to statistical fluctuations these functions may exhibit cross-plane coupling which is absent in the underlying dynamical system so that they should be used only for the normal mode identification and putting in the canonical order (x, y, t).

Besides the beta-functions the effective dispersion functions can also be estimated as described in [3]. Requiring the 3rd mode projection on time coordinate to be zero and taking ratio of its x-projection to δ-projection we get

$$D_x = \frac{x}{\delta} = \frac{\mathrm{Im}[(\underline{v}_5^*)_5(\underline{v}_5)_1]}{\mathrm{Im}[(\underline{v}_5^*)_5(\underline{v}_5)_6]} \tag{15}$$

and similarly for D_y with index replacement $1\rightarrow3$.

An example of application of the described algorithm for the analysis of the ionization cooling of muons can be found in [4]. It is also planned to use it in MADX-SC code for fitting the distribution of tracking particles [5].

APPENDIX

We can use the described above method for computation of eigen-emittances to answer the often raised question of how important it is to use canonical momenta instead of mechanical momenta. Let us assume that we have a distribution of particles in axisymmetric magnetic field B_z such that

$$<x^2>=<y^2>=\sigma^2, \quad <P_x^2>=<P_y^2>=\sigma_p^2, \tag{16}$$

with all other elements of the Σ-matrix being zero (we consider here a 4D case). Obviously both normal modes have equal emittances $\varepsilon_0 = \sigma_p\sigma$ and beta-functions $\beta_\perp = \sigma/\sigma_p$.

Now, if we would use the mechanical momenta, the non-zero covariance coefficients would be

$$\begin{aligned} &<x^2>=<y^2>=\sigma^2, \\ &<p_x^2>=<P_x^2>+K^2<y^2>=\sigma_p^2+K^2\sigma^2=<p_y^2>, \\ &<p_x y>=-<p_y x>=K\sigma^2, \quad K=B_z/2B\rho \end{aligned} \tag{17}$$

we would obtain for eigen-emittances

$$\varepsilon_{1,2}^2 = \varepsilon_0^2(1+2K^2\beta_\perp^2 \pm 2|K|\beta_\perp\sqrt{1+K^2\beta_\perp^2}) \tag{18}$$

Dependence of eigen-emittances on parameter $K\beta_\perp$ is shown in Fig. 2. For the matched beta-function value $K\beta_\perp = 1$ emittances (18) differ by more than a factor of two from the correct values. However their product – the 4D emittance – remains correct: $\varepsilon_1\varepsilon_2=\varepsilon_0^2$.

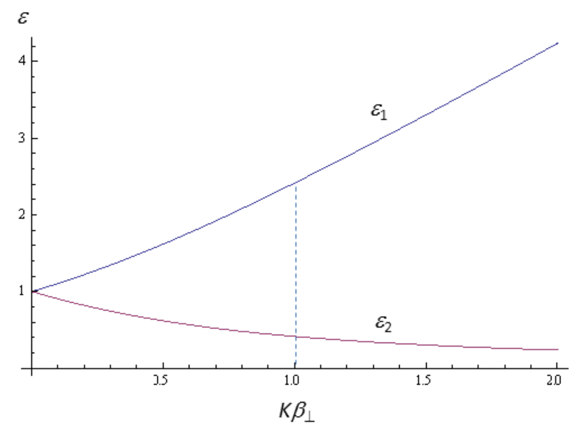

Figure 2: Eigen-emittances with mechanical momenta used vs. normalized magnetic field strength.

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