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CONSIDERATIONS AND FINDINGS ON BEAM VORTICITY DYNAMICS

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Abstract

Rotation of beams is usually quantified through angular momentum rather than through vorticity. However, the difference of the two transverse eigen-emittances is linked more strongly to vorticity than to angular momentum. It has been found that the dynamics of vorticity has remarkable similarity to the dynamics of the beam envelope along channels of solenoids and quadrupole triplets. Transport matrices of vorticity, corresponding phase advances, and Twiss parameters look very similar and are partially even identical to their counterparts concerning envelopes. Corresponding to emittance, the quantity of vortissance, being a constant of motion, is defined. Unlike emittance, for vorticity-dominated beams, it may take imaginary values causing imaginary Twiss parameters and negative or zero phase advances along a finite beam line section.

ANGULAR MOMENTUM AND VORTICITY

The rms angular momentum of a particle beam is defined through the beam's second moments as

$$L := \langle xy' \rangle - \langle x'y \rangle . \quad (1)$$

Usually, the angular momentum is regarded as the quantification of rotation being justified by its outstanding role in physics. Nevertheless, other quantities may serve for this purpose as well, among them for instance the beam vorticity

$$\mathcal{V} = A \int_A \left[\vec{\nabla} \times \vec{r}\,' \right] \cdot d\vec{A} , \quad (2)$$

obtained from the rotation of the derivative of the particle beam position and the beam rms area

$$A^2 := \langle x^2 \rangle \langle y^2 \rangle - \langle xy \rangle^2 . \quad (3)$$

The above expression for vorticity may be re-phrased [1] as

$$\mathcal{V} = \langle y^2 \rangle \langle xy' \rangle - \langle x^2 \rangle \langle yx' \rangle + \langle xy \rangle (\langle xx' \rangle - \langle yy' \rangle) . \quad (4)$$

Figure 1 illustrates the difference between angular momentum and vorticity using an ellipse performing two types of rotation.

Determining the angular momentum and $\mathcal{V}/A$ reveals for the rigid rotation [1]

$$L_{rig} = \frac{\omega}{4} a^2 (1 + r^2) , \quad (5)$$

$$(\mathcal{V}/A)_{rig} = \frac{\omega}{2} a^2 r , \quad (6)$$

while for the intrinsic rotation one obtains

$$L_{int} = \frac{\omega}{2} a^2 r , \quad (7)$$

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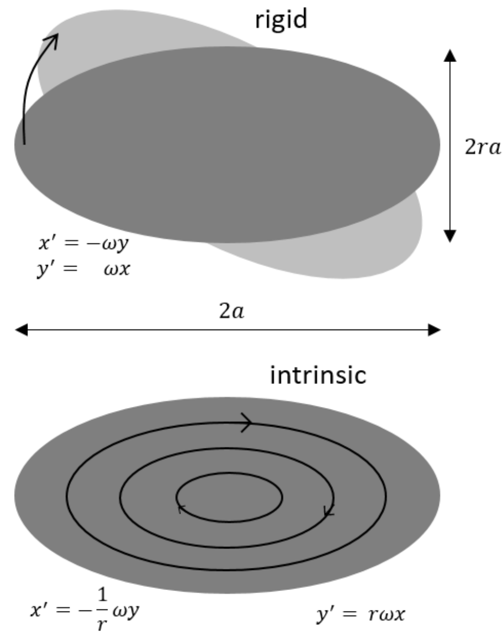

Figure 1: Ellipse with aspect ratio r performing a rigid rotation (upper) and an intrinsic rotation (lower).

$$(\mathcal{V}/A)_{int} = \frac{\omega}{4} a^2 (1 + r^2) . \quad (8)$$

The expressions for L and $\mathcal{V}/A$ flip when flipping from rigid to intrinsic rotation. Additionally, for extreme aspect ratios of $r \ll 1$ or $r \gg 1$, the rigid rotation has just angular momentum and vanishing V/A (relatively), while the intrinsic rotation has just V/A but vanishing angular momentum (relatively). Another special case is the circle ($r = 1$) and just for objects with cylindrical symmetry, angular momentum is equal to vorticity

$$L = \mathcal{V}/A . \quad (9)$$

Vorticity enters into beam dynamics through its tight relation to the two transverse eigen-emittances ε_1 and ε_2 [2] of a beam. The latter are equal to the two projected rms emittances, if and only if all correlations between the two transverse planes have been removed to zero. Removing them preserves the eigen-emittances and changes the projected emittances.

Transverse beam eigen-emittances are calculated through beam rms moments as

$$\varepsilon_{1/2} = \frac{1}{2} \sqrt{-tr[(CJ)^2] \pm \sqrt{tr^2[(CJ)^2] - 16\, det\,(C)}} , \quad (10)$$

with

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$$C = \begin{bmatrix} \langle x^2\rangle & \langle xx'\rangle & \langle xy\rangle & \langle xy'\rangle \\ \langle xx'\rangle & \langle x'^2\rangle & \langle yx'\rangle & \langle x'y'\rangle \\ \langle xy\rangle & \langle yx'\rangle & \langle y^2\rangle & \langle yy'\rangle \\ \langle xy'\rangle & \langle x'y'\rangle & \langle yy'\rangle & \langle y'^2\rangle \end{bmatrix}, \tag{11}$$

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \tag{12}$$

and $\varepsilon_{4d} := (\det C)^{\frac{1}{2}} = \varepsilon_1 \cdot \varepsilon_2$.

The eigen-emittances of the two ellipses are calculated as [1]

$$\epsilon_{1,rig} = (\mathcal{V}/A)_{rig} \leq L_{rig}\,, \tag{13}$$

$$\epsilon_{2,rig} = 0 \tag{14}$$

and

$$\epsilon_{1,int} = (\mathcal{V}/A)_{int} \geq L_{int}\,, \tag{15}$$

$$\epsilon_{2,int} = 0\,. \tag{16}$$

These relations are one example illustrating that eigen-emittances are related to vorticity rather than to angular momentum.

In general, for short beam line sections along which the rms beam area A remains fairly constant, the tight relation is given by [1]

$$(\varepsilon_1 - \varepsilon_2)^2 - (\mathcal{V}/A)^2 = const.\,, \tag{17}$$

stating that the difference of eigen-emittances changes with the change of beam vorticity rather than with the change of angular momentum. Short skewed quadrupoles change the beam angular momentum but they leave constant both eigen-emittances. In turn, a beam passing the entrance or exit fringe field of a solenoid, experiences change of its vorticity and change of its eigen-emittances in accordance with Eq. (17) [1]. In the following, it will be shown that apart from this relation, transformation of vorticity along a beam line from solenoids has remarkable similarity to the transformation of the beam envelope. A detailed treatment of the following sections is given in [3].

TRANSFORMATION OF BEAM ENVELOPE AND VORTICITY

Defining the horizontal envelope vector

$$\vec{X} := [\langle x^2\rangle, \langle x^2\rangle', \langle x^2\rangle''] \tag{18}$$

and transporting it along a simple drift of length d gives

$$\vec{X} \to \begin{bmatrix} 1 & d & \frac{d^2}{2} \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix} \cdot \vec{X}\,. \tag{19}$$

Using a horizontally focusing quadrupole of effective length L_q and gradient G, results into

$$\vec{X} \to \begin{bmatrix} C_q^2 & C_qS_q/u & S_q^2/(2u^2) \\ -2uC_qS_q & C_q^2 - S_q^2 & C_qS_q/u \\ 2u^2S_q^2 & -2uC_qS_q & C_q^2 \end{bmatrix} \cdot \vec{X}\,, \tag{20}$$

where C_q:=cos(Ω), S_q:=sin(Ω), $\Omega := uL_q$, $u := |G/(B\rho)|^{\frac{1}{2}}$, and $(B\rho)$ is the beam rigidity.

Defining the vorticity vector

$$\vec{\mathcal{V}} := [\mathcal{V}, \mathcal{V}', \mathcal{V}''] \tag{21}$$

being transported through a drift delivers the transformation

$$\vec{\mathcal{V}} \to \begin{bmatrix} 1 & d & \frac{d^2}{2} \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix} \cdot \vec{\mathcal{V}}\,, \tag{22}$$

which looks identical to the respective transformation of the beam envelope. Transportation of the vorticity vector through a solenoid of effective length L_s and magnetic field strength B reveals

$$\vec{\mathcal{V}} \to \begin{bmatrix} C^2 & CS/\kappa & S^2/(2\kappa^2) \\ -2\kappa CS & C^2 - S^2 & CS/\kappa \\ 2\kappa^2S^2 & -2\kappa CS & C^2 \end{bmatrix} \cdot \vec{\mathcal{V}}\,, \tag{23}$$

where C:=cos(ω), S:=sin(ω), $\omega := \kappa L_s$, and $2\kappa := B/(B\rho)$. There is strong similarity between Eqs. (20) and (23). The beam envelope is transformed by a focusing quadrupole as the beam vorticity is transformed by a solenoid.

TWISS PARAMETERS

The similarities of transformation of envelope and vorticity extend towards the Twiss parameters. In analogy to the horizontal rms emittance

$$\epsilon_x = \sqrt{\langle x^2\rangle\frac{\langle x^2\rangle''}{2} - \frac{\langle x^2\rangle'^2}{4}}\,, \tag{24}$$

being preserved along drifts and regular quadrupoles, the "rms vortissance" shall be defined as

$$V := \sqrt{\mathcal{V}\frac{\mathcal{V}''}{2} - \frac{\mathcal{V}'^2}{4}}\,. \tag{25}$$

Vorticity Twiss parameters are defined as their envelope counterparts through

$$\beta_v := \frac{\mathcal{V}}{V}\,, \tag{26}$$

$$\alpha_v := -\frac{\beta_v'}{2} = \frac{-\mathcal{V}'}{2V}\,, \tag{27}$$

and

$$\gamma_v := \frac{1+\alpha_v^2}{\beta_v} = \frac{\mathcal{V}''}{2V}\,. \tag{28}$$

Vorticity betatron phase advances between two locations a and b along s may be defined as for the envelope through

$$\Delta\Phi_{v,a,b} := \int_a^b \frac{ds}{\beta_v(s)}\,. \tag{29}$$

Unlike the envelope emittance and Twiss parameters, the vortissance and its Twiss parameters may assume purely imaginary values. Vorticity Twiss parameters can take also negative values (real or imaginary). This has impact on the corresponding phase advances as well as on the characteristics of periodic vorticity solutions.

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Real Vortissance

If $V^2 > 0$, the vortissance is real as well as its Twiss parameters. However, the vorticity $\mathcal{V}$ may be negative and accordingly, the Twiss parameters are negative as well. Physically, this reflects just a beam rotating clock or counter-clockwise. Apart from this ambiguity in sign, there is full equivalence of vorticity dynamics along lines from solenoids and envelope dynamics along lines from focusing quadrupoles. As for envelope Twiss parameters, the vorticity Twiss parameters cannot change their signs. The constance of $V^2 > 0$ enforces both, $\mathcal{V}$ and $\mathcal{V}''$, to be of same sign and being different from zero. Hence, $V^2 > 0$ imposes an intrinsic defocusing of $\mathcal{V}$ and $\mathcal{V}''$ away from zero. The sign of both is preserved, i.e., the zeros cannot be crossed. In envelope dynamics, this is the well known emittance defocusing term.

Imaginary Vortissance

In case of $V^2 < 0$, the Twiss parameters are purely imaginary and shall be calculated as

$$\beta_v \ := -i\,\frac{\mathcal{V}}{|V|}\,, \tag{30}$$

$$\alpha_v \ := -\frac{\beta_v'}{2} \ = i\,\frac{\mathcal{V}'}{2|V|}\,, \tag{31}$$

$$\gamma_v \ := \ \frac{1+\alpha_v^2}{\beta_v} \ = -i\frac{\mathcal{V}''}{2|V|}. \tag{32}$$

As for real vortissance, the Twiss parameters can take negative (but imaginary) values. Phase advances are purely imaginary and can take negative values as well. In contrast to real vortissance, there is no intrinsic defocusing of vorticity. Accordingly, $\mathcal{V}$ and/or $\mathcal{V}''$ as well as the Twiss parameters β_v and/or γ_v may be zero, or different in sign.

However, $\mathcal{V} = \beta_v = 0$ does not cause an ill-defined γ_v, since the constance of $V^2 < 0$ imposes $\mathcal{V}' = 2iV$ and hence $\alpha_v^2 = -1$, thus preventing the singularity of γ_v. Additionally, the singularity is physically prevented by $\gamma_v = \mathcal{V}''/2V$ with $V \neq 0$.

Eventual zero crossings of β_v do not harm determination of the imaginary phase advance according to Eq. (29). But they may result into zero phase advance between two locations. This does not occur in envelope dynamics nor in vorticity dynamics at real vortissance. The possibility of zero crossings of vorticity and of vanishing phase advances has impact on the nature of periodic solutions. Details on these aspects are reported in [3].

VORTICITY TWISS PARAMETERS TRANSPORT MATRICES

As in envelope dynamics, the vorticity Twiss parameters can be written into a matrix

$$T_v \ := \begin{bmatrix} \beta_v & -\alpha_v \\ -\alpha_v & \beta_v \end{bmatrix}, \tag{33}$$

which then is transported through a given lattice element. Transport through a drift reads as

$$T_{v,f} \ := \ M_t \cdot T_{v,i} \cdot M_t^T\,, \tag{34}$$

with

$$M_{t,d} \ = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \tag{35}$$

being identical to the transformation of envelope Twiss parameters through a drift. Transport of vorticity Twiss parameters through a solenoid is expressed by the matrix

$$M_{t,s} \ = \begin{bmatrix} C & S/\kappa \\ -\kappa S & C \end{bmatrix} \tag{36}$$

which is in full analogy to the transport of envelope Twiss parameters through a focusing quadrupole with the matrix

$$\mathcal{M}_{t,q} \ = \begin{bmatrix} C_q & S_q/u \\ -uS_q & C_q \end{bmatrix}. \tag{37}$$

Total amounts of vorticity phase advances between two positions a and b are defined by the traces of the vorticity transport matrices as

$$|\Delta\Phi_{v,a,b}| \ := \int_a^b \frac{ds}{\beta_v(s)} \ = \ \frac{1}{2}Tr\,[M_{t,a\to b}]\,. \tag{38}$$

Sources of Vortissance

Transformation of vorticity by quadrupoles and by solenoid fringe fields cannot be expressed by symplectic matrices. In fact, these two beam line elements are sources (or sinks) of vortissance. For instance, a beam with just diagonal moments being different from zero and passing a solenoid fringe field will acquire real vortissance of

$$V \ = \ 2\kappa A\sqrt{\varepsilon_1^2+\varepsilon_2^2+2\kappa^2A^2}\,. \tag{39}$$

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