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MATCHED TRANSPORT OF INTENSE AND COASTING BEAMS THROUGH QUADRUPOLE CHANNELS

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Abstract

Imposing angular momentum to a particle beam increases its stability against perturbations from space charge. In order to fully explore this potential, proper matching of intense coupled beams along regular lattices is mandatory. Herein, a novel procedure assuring matched transport is described and benchmarked through simulations. The concept of matched transport along periodic lattices has been extended from uncoupled beams to those with considerable coupling between the two transverse degrees of freedom. For coupled beams, matching means the extension of cell-to-cell periodicity from just transverse envelopes to the coupled beam moments and to quantities being derived from these.

INTRODUCTION

Preservation of beam quality is of major concern for acceleration and transport especially of intense hadron beams. This aim is reached at best through provision of smooth and periodic beam envelopes, being so-called matched to the periodicity of the external focusing lattice. For the time being, the quality of matching has been evaluated through the periodicity of spatial beam envelopes. This is fully sufficient as long as there is no coupling between the phase space planes (for brevity “planes”), neither in beam properties nor in lattice properties.

The TRACE-2D code [1] is well suited to provide for a matching beam line between a given initial beam matrix and a desired exit beam matrix even for a full 4D scenario. However, it is an intrinsic property of the periodic-solution problem that the initial beam matrix at the entrance of the periodic channel is unknown. Accordingly, this code cannot be applied to the present scenario in a straightforward way. This paper aims to demonstrate that a 4D-periodic cell-by-cell solution exists and demonstrates its derivation [2].

Coupled beams inhabit ten independent second-order rms moments. They are summarized within the symmetric beam moments matrix

$$C := \begin{bmatrix} \langle xx\rangle & \langle xx'\rangle & \langle xy\rangle & \langle xy'\rangle \\ \langle x'x\rangle & \langle x'x'\rangle & \langle x'y\rangle & \langle x'y'\rangle \\ \langle yx\rangle & \langle yx'\rangle & \langle yy\rangle & \langle yy'\rangle \\ \langle y'x\rangle & \langle y'x'\rangle & \langle y'y\rangle & \langle y'y'\rangle \end{bmatrix}, \tag{1}$$

and four of its elements quantify beam coupling. Beams are x-y coupled if at least one of these elements is different from zero.

The beam line being used to determine the periodic solution of an intense coupled beam along a periodic channel is sketched systematically in Fig. 1.

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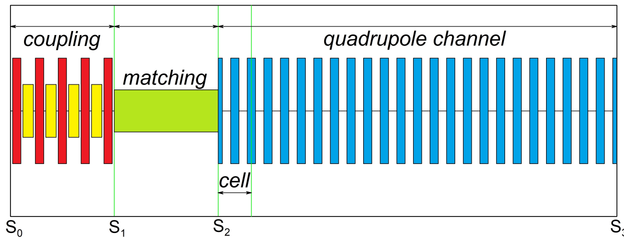

Figure 1: The beam line comprises three parts: (I) coupling production section; (II) matching section; (III) regular quadrupole doublet section (twelve cells). Space charge effects are not considered along the first two sections (see text).

At the beginning of the beam line, an uncoupled beam is assumed with beam sigma-matrix $C(s_0)$. The beam matrix at the beginning of the matching section is

$$C(s_1) = \wp \cdot C(s_0) \cdot \wp^{\mathrm{T}}, \tag{2}$$

and $\wp$ indicates the transfer matrix of the coupling section.

In order to obtain a periodic solution for this coupled beam, the details of the matching section are not required as seen in the following. However, it is modeled by a transport matrix including 16 elements (in units of m and rad)

$$\mathfrak{R}(m_1, m_2, \ldots, m_{16}) = \begin{bmatrix} m_1 & m_2 & m_3 & m_4 \\ m_5 & m_6 & m_7 & m_8 \\ m_9 & m_{10} & m_{11} & m_{12} \\ m_{13} & m_{14} & m_{15} & m_{16} \end{bmatrix}. \tag{3}$$

Although initially being unknown, the 16 elements must ensure that $\mathfrak{R}$ is symplectic. For brevity, the set of $m_1, m_2, \ldots, m_{16}$ shall be denoted by $\aleph$.

MODELLING OF PERIODIC CHANNEL

For zero current, the effective focusing forces are given solely by the external lattice. The actual beam shape has no influence on them and therefore the periodic solution even for coupled beams may be found analytically. For intense beams instead, defocusing space charge forces depend on the beam shape and orientation in real space. Actually, they depend also on the spatial distribution. However, since modelling of space charge forces using rms-equivalent KV-distributions proofed to work very well for matching purposes, this approach is followed here as well.

The periodic solution (zero current) meets the condition

$$C(s_2) = \mathfrak{I} \cdot C(s_2) \cdot \mathfrak{I}^{\mathrm{T}}, \tag{4}$$

and the transport matrix from the exit of the coupling section s_1 to the exit of the first cell is

$$\mho(\aleph) = \mathfrak{I} \cdot \mathfrak{R}(\aleph), \tag{5}$$

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where $\mathfrak{I}$ is fully known from the cell of the quadrupole channel.

From first principles, neither the periodic solution is known nor are the elements $\aleph$ that provide for the according matching from s_1 to the entrance of the channel s_2. The iterative procedure to obtain finally both, starts with a guessed initial set $\aleph^i$ that just meets the condition of being symplectic. It will most likely not meet the condition of the periodic solution, i.e.,

$$\Re\left(\aleph^i\right) \cdot C\left(s_1\right) \cdot \Re^{\mathrm{T}}\left(\aleph^i\right) \neq \mho\left(\aleph^i\right) \cdot C\left(s_1\right) \cdot \mho^{\mathrm{T}}\left(\aleph^i\right) , \quad (6)$$

hence the beam matrix in front of the channel is different from the one behind the first cell.

With the MATHCAD [3] routine *Minerr*, a set of matching matrix elements $\aleph^0$ for zero beam current can be found, such that the symplectic condition is met sharply together with providing periodicity. The routine is dedicated to solve an under-determined system of equations with a defined set boundary conditions, such that

$$\Re\left(\aleph^0\right) \cdot C\left(s_1\right) \cdot \Re^{\mathrm{T}}\left(\aleph^0\right) = \mho\left(\aleph^0\right) \cdot C\left(s_1\right) \cdot \mho^{\mathrm{T}}\left(\aleph^0\right) . \quad (7)$$

PERIODIC SOLUTION WITH CURRENT

The iterative procedure starts from the beam moments matrix $C\left(s_1\right)$ being then transported through the matching line $\Re\left(\aleph^0\right)$ for zero current. The resulting beam matrix at the entrance to the channel

$$C^0\left(s_2\right) = \Re\left(\aleph^0\right) \cdot C\left(s_1\right) \cdot \Re^{\mathrm{T}}\left(\aleph^0\right) , \quad (8)$$

is then tracked with high current (10 mA) through one cell. Accordingly, the total transport matrix of the cell $\mathfrak{I}_{\mathrm{sc}}\left(\aleph^0\right)$ is a result of the tracking procedure for high current. $\mathfrak{I}_{\mathrm{sc}}\left(\aleph^0\right)$ depends on the current I and on the spatial beam parameters at the entrance of the channel. The 4×4 elements of $\mathfrak{I}_{\mathrm{sc}}\left(\aleph^0\right)$ are stored for further use. Most likely, $C^0\left(s_2\right)$ does not meet the condition of the periodic solution with current, i.e,

$$C^0\left(s_2\right) \neq \mathfrak{I}_{\mathrm{sc}}\left(\aleph^0\right) \cdot \Re\left(\aleph^0\right) \cdot C\left(s_1\right) \cdot \Re^{\mathrm{T}}\left(\aleph^0\right) \cdot \mathfrak{I}_{\mathrm{sc}}^{\mathrm{T}}\left(\aleph^0\right) . \quad (9)$$

However, the cell matrix $\mathfrak{I}_{\mathrm{sc}}\left(\aleph^0\right)$ is used to re-adapt the matching setting such, that a new matching $\aleph^1$ is found which provides for equal beam matrices before and after transport through the cell matrix $\mathfrak{I}_{\mathrm{sc}}\left(\aleph^0\right)$

$$C^1\left(s_2\right) = \mathfrak{I}_{\mathrm{sc}}\left(\aleph^0\right) \cdot \Re\left(\aleph^1\right) \cdot C\left(s_1\right) \cdot \Re^{\mathrm{T}}\left(\aleph^1\right) \cdot \mathfrak{I}_{\mathrm{sc}}^{\mathrm{T}}\left(\aleph^0\right) , \quad (10)$$

emphasizing that the above equation uses the stored elements of $\mathfrak{I}_{\mathrm{sc}}\left(\aleph^0\right)$.

This new matching $\aleph^1$ delivers the beam matrix $C^1\left(s_2\right)$ in front of the channel. It is now re-tracked with current through the cell. The tracking will provide a new cell matrix $\mathfrak{I}_{\mathrm{sc}}\left(\aleph^1\right)$. Again its 4×4 elements are stored to re-adapt the matching to a setting $\aleph^2$ meeting the periodic solution assuming the new matrix $\mathfrak{I}_{\mathrm{sc}}\left(\aleph^1\right)$ along the channel

$$C^2\left(s_2\right) = \mathfrak{I}_{\mathrm{sc}}\left(\aleph^1\right) \cdot \Re\left(\aleph^2\right) \cdot C\left(s_1\right) \cdot \Re^{\mathrm{T}}\left(\aleph^2\right) \cdot \mathfrak{I}_{\mathrm{sc}}^{\mathrm{T}}\left(\aleph^1\right) . \quad (11)$$

This in turn provides a new beam matrix $C^2\left(s_2\right)$ in front of the channel, which changes the transport matrix of the cell to $\mathfrak{I}_{\mathrm{sc}}\left(\aleph^2\right)$. Continuing this procedure finally converges, i.e., the changes from $\aleph^{n-1}$ to $\aleph^n$ become very small and finally negligible. Accordingly, after a sufficient amount of iterations j, the periodic condition is fulfilled through

$$C^j\left(s_2\right) \approx \mathfrak{I}_{\mathrm{sc}}\left(\aleph^j\right) \cdot \Re\left(\aleph^j\right) \cdot C\left(s_1\right) \cdot \Re^{\mathrm{T}}\left(\aleph^j\right) \cdot \mathfrak{I}_{\mathrm{sc}}^{\mathrm{T}}\left(\aleph^j\right) . \quad (12)$$

The matrix $C^j\left(s_2\right)$ contains the periodic beam moments at the entrance to the channel and $\mathfrak{I}_{\mathrm{sc}}\left(\aleph^j\right)$ is the periodic transport matrix of the cell including current and coupling.

In case of the example presented here, sufficient convergence has been reached at $j = 6$ and the corresponding beam matrix (in units of mm and mrad) is

$$C^6\left(s_2\right) = \begin{bmatrix} +147.8 & +0.006 & +59.34 & -0.006 \\ \cdots & +80.44 & +0.006 & +114.9 \\ \cdots & \cdots & +47.36 & +0.011 \\ \cdots & \cdots & \cdots & +286.7 \end{bmatrix} , \quad (13)$$

The corresponding output beam matrix (in units of mm and mrad) is

$$C^6\left(s_2 + \ell\right) = \begin{bmatrix} +147.8 & +0.026 & +59.33 & -0.106 \\ \cdots & +80.41 & +0.035 & +114.9 \\ \cdots & \cdots & +47.34 & -0.017 \\ \cdots & \cdots & \cdots & +286.8 \end{bmatrix} . \quad (14)$$

This section shall be closed by a comparison of the fully 4D-periodic solution along the channel with the one obtained from simple 2D-envelope matching. Figure 2 plots the six 2D projections of the phase space ellipses in front of and behind the first cell of the periodic channel. It has been shown that cell-to-cell periodicity of an intense coupled coasting beam can be achieved under the assumption of a KV-distribution.

The so-called 2D-envelope matching [4] ignores the coupled beam moments leading to the non-coupling matching transfer matrix and the resulting beam matrix at the entrance of the channel is (in units of mm and mrad)

$$C^{\dagger}\left(s_2\right) = \begin{bmatrix} +183.6 & +0.169 & +100.0 & +0.000 \\ \cdots & +101.1 & +0.000 & -10.00 \\ \cdots & \cdots & +59.34 & +0.011 \\ \cdots & \cdots & \cdots & +379.7 \end{bmatrix} , \quad (15)$$

while the corresponding beam matrix at the exit is (in units of mm and mrad)

$$C^{\dagger}\left(s_2 + \ell\right) = \begin{bmatrix} +186.1 & +1.967 & +6.026 & -103.7 \\ \cdots & +105.1 & -30.71 & +157.4 \\ \cdots & \cdots & +61.88 & +5.640 \\ \cdots & \cdots & \cdots & +388.7 \end{bmatrix} . \quad (16)$$

Figure 3 compares the six 2D projections of the 4D phase in front of and behind the first cell of the periodic channel. As expected, periodicity is achieved for the horizontal and vertical planes. However, there is no periodicity in the projections that mix the two planes. In the following section, the results from KV-rms-tracking are benchmarked with particle tracking of a beam with Gaussian distribution.

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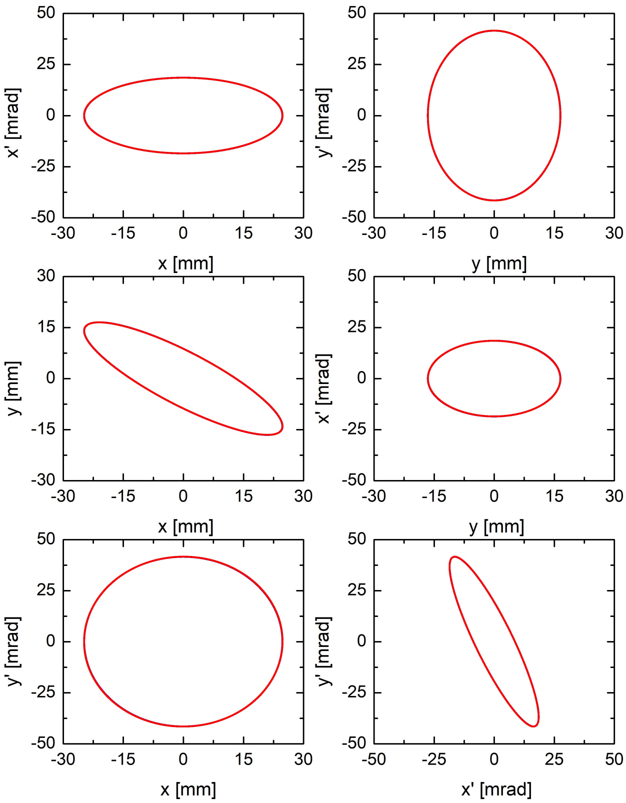

Figure 2: From full 4D-periodic solution: projected 4×rms ellipses of the beam second moments matrix at the entrance (blue) and exit (red) of the periodic channel for a coupled proton beam with 10 mA. It is obtained that $C^6(s_2) \approx C^6(s_2+\ell)$.

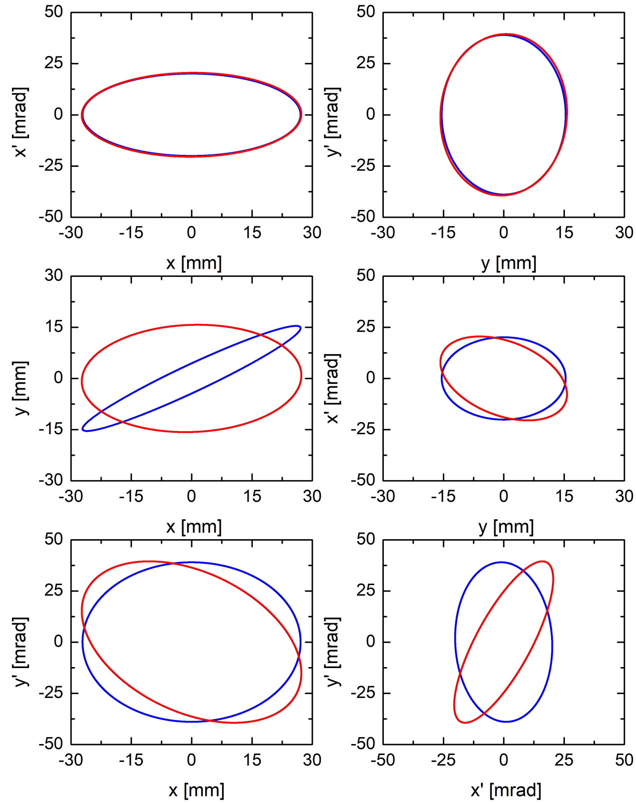

Figure 3: From simple 2D-envelope matching: projected 4×rms ellipses of the beam second moments matrix at the entrance (blue) and exit (red) of the periodic channel for a coupled proton beam with 10 mA. It is obtained that $C^\dagger(s_2) \neq C^\dagger(s_2+\ell)$.

BENCHMARKING

Benchmarking has been done with MATHCAD using a KV-type beam and BEAMPATH [5] using a Gaussian-type beam. Initial distributions of 2×10^4 particles are rms equivalent to the second beam moments matrices $C^6(s_2)$ and $C^\dagger(s_2)$, respectively. Particle-tracking simulations have been done using a 10 mA proton beam and 12 cells of the periodic channel. Figures 4 and 5 show the transverse 2×rms-beam sizes along the quadrupole channel obtained from rms tracking with mathcad and extracted from particle tracking simulation with BEAMPATH.

Applying cell-to-cell second moments matching, both, transverse 2×rms-beam sizes from KV-rms tracking and from particle tracking a Gaussian beam, reveal a high degree of envelope matching to the lattice periodicity. The KV-based rms-beam size is very regular and the Gaussian rms-beam size shows slight fluctuation. Those are to be expected since space charge forces especially at the outer parts of the beam are different for KV and for Gaussian distributions. The matching proofed to work very well even for the Gaussian beam.

Applying simple 2D-envelopes matching, the transverse 2×rms-beam sizes are still well matched to the periodic quadrupole channel, although the fluctuations are notably larger compared to those of the full 4D solution.

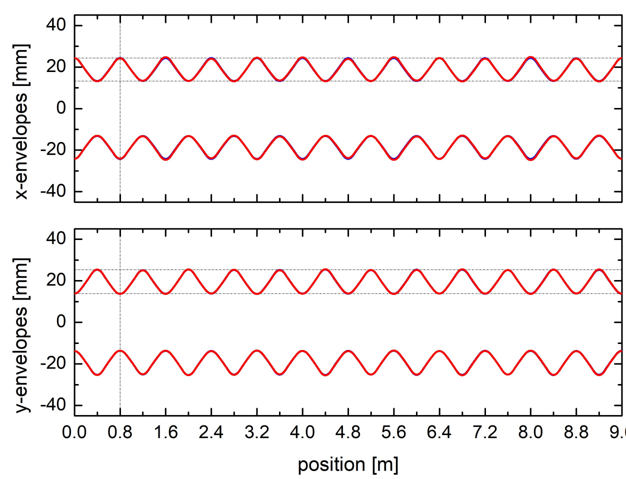

Figure 4: From full 4D-periodic solution: horizontal and vertical 2×rms-beam sizes of a coupled 10 mA proton beam along a regular FODO quadrupole channel as obtained from rms tracking (blue) and particle tracking (red). The initial particle distribution is rms equivalent to matrix $C^6(s_2)$.

Eigen-emittances are preserved by symplectic transformations as KV-rms tracking. Instead, nonlinear space charge forces occurring at particle tracking of a Gaussian beam do not preserve the eigen-emittances. Figures 6 and 7 plot

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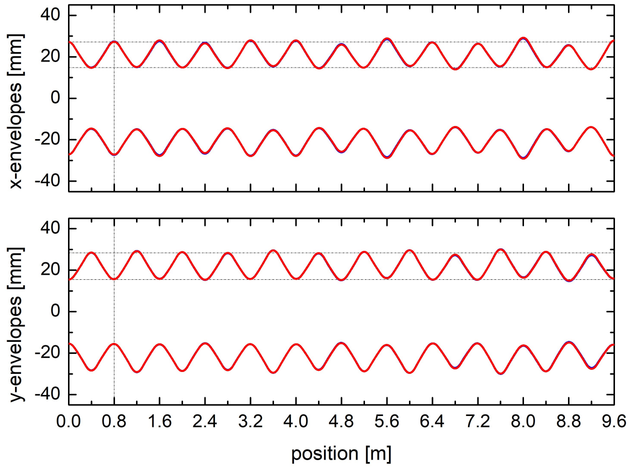

Figure 5: From simple 2D-envelope matching: horizontal and vertical 2×rms-beam sizes of a coupled 10 mA proton beam along a regular FODO quadrupole channel as obtained from rms tracking (blue) and particle tracking (red). The initial particle distribution is rms equivalent to matrix $C^{\dagger}\,(s_2)$.

eigen-emittances, projected rms emittances, and square roots of 4D emittances along the channel from particle-tracking simulations.

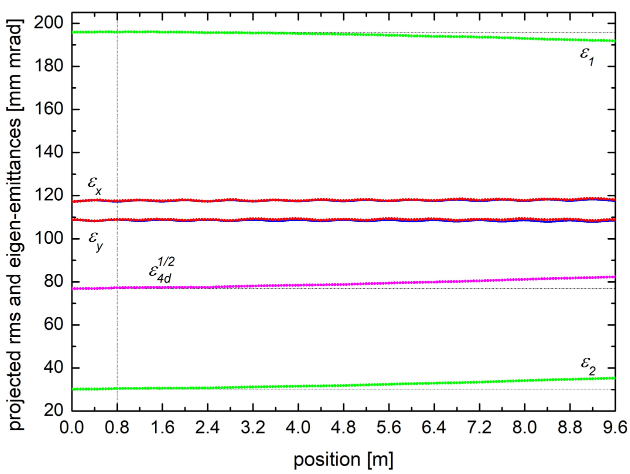

Figure 6: From full 4D-periodic solution: transverse projected rms emittances as obtained from rms tracking (blue) and particle tracking (red). Green (magenta) curves indicate the eigen-emittances (square roots of the 4D emittances) calculated from particle tracking. The initial particle distribution is rms equivalent to beam matrix $C^6\,(s_2)$.

CONCLUSION

It has been shown that cell-to-cell 4D-matching can be achieved for a coupled beam with considerable space charge forces. This has been accomplished by rms-tracking of coupled beams with KV-distribution combined with a dedicated iterative procedure of tracking and re-matching.

Benchmarking with an initial Gaussian distribution along a channel with a large cell number revealed that the method works very well.

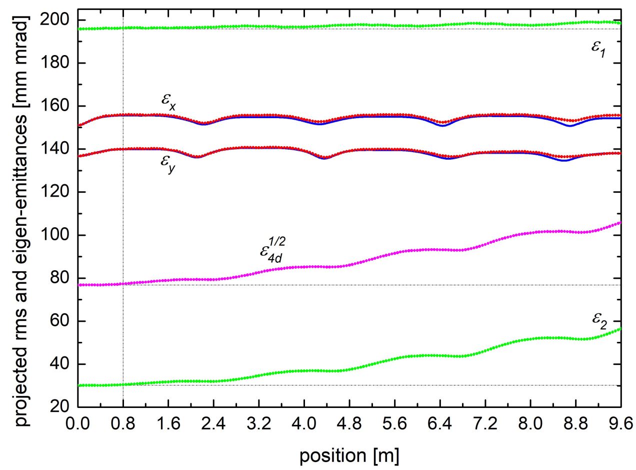

Figure 7: From simple 2D-envelope matching: transverse projected rms emittances as obtained from rms tracking (blue) and particle tracking (red). Green (magenta) curves indicate the eigen-emittances (square roots of the 4D emittances) calculated from particle tracking. The initial particle distribution is rms equivalent to beam matrix $C^{\dagger}\,(s_2)$.

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