MODELING THE DYNAMICS OF CHARGED-PARTICLE BEAMS USING A FOKKER-PLANCK APPROACH*

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Abstract

At the previous Linac Conference, we introduced a semianalytic Fokker-Planck formalism for calculating the evolution of intense, nonrelativistic, mismatched beams propagating through focusing channels. We have since elaborated on its physical basis and greatly expanded its applicability. In this paper, we implement the model to study the dynamics of a circularly symmetric beam propagating through a linear focusing channel. An example is discussed for an ion beam and accelerator parameters which are representative of high-current spallation neutron sources in which space-charge forces are important. The example illustrates the dynamics of emittance growth and halo formation in the beam.

Introduction

A space-charge-dominated beam which is rms-mismatched upon injection into an acceleration stage undergoes a very complicated evolutionary sequence. As a consequence of mismatch the beam relaxes to a state of quasi-equilibrium on a time scale which can be short compared to the acceleration time, and which is much shorter than the collisional relaxation time. The relaxation is a collisionless process akin to violent relaxation in stellar systems, and it is due to the excitation of a turbulent spectrum of electrostatic fluctuations. This spectrum is triggered almost immediately, *i.e.*, after roughly one-quarter of the beam's plasma period, and it then contains the free energy associated with the mismatch. As particles interact with the rapidly fluctuating global and local self-fields of the beam, their energies are not conserved. Instead, the particles gradually consume the free energy, thereby dissipating the turbulent spectrum. This results in an overall "heating" of the beam, with some particles being ejected into large-amplitude orbits. Thus, the net effects of violent relaxation are emittance growth and halo formation.

In earlier papers we introduced a semianalytic Fokker-Planck-Poisson (FPP) formalism with which to calculate the rapidly evolving beam properties as functions of time [1-3]. In particular, Ref. [3] provides a detailed account of the evolutionary sequence, the justification for the FPP approach, and the method for solving the governing equations. The basic idea is to decompose the distribution function of particles in the phase-space of a single particle into a coarse-grained component and localized fluctuations. The fluctuation spectrum generates dynamical friction and diffusion in velocity space which in turn drive the coarse-grained distribution toward Maxwell-Boltzmann equilibrium. Thus, FPP incorporates a working hypothesis that the quasi-stationary state resulting from violent relaxation is Maxwell-Boltzmann.

FPP, being a statistical theory, accounts for the orbits of all of the particles comprising the beam. Thus, in principle, FPP also accounts for the totality of the mode spectrum, *i.e.*, for both systematic global oscillations and stochastic local fluctuations. It nevertheless is instructive to consider separately the effects of the global and local modes, as has been the trend in the recent literature. Accordingly, in what follows, we shall present results of an elementary calculation of the time scale for particle ejection via oscillating-core/single-particle interactions, and then we shall apply the FPP formalism to calculate the dynamics of emittance growth and halo formation in a beam which is representative of high-current spallation neutron sources.

Oscillating-Core/Single-Particle Dynamics

The time scale t_{cp} for ejection of particles having orbital periods in resonance with the core-oscillation period t_c can be written in the form

$$t_{cp} \sim t_c \frac{\Delta U}{\Delta E_p},$$

where ΔU is the increase in the particle's potential energy measured at the turning points in its orbit, and ΔE_p is the energy the particle gains each orbital period. If the core is modeled as a uniform cylindrical beam of radius a, then the time scale for the resonant particle to climb up the potential well from $U(r{=}a)$ to $U(r{=}2a)$ is [3]

$$t_{cp} \sim t_c \frac{3-(1-\kappa^2)2ln2}{(1-\kappa^2)F(2\rho_0^{-1}-1)},$$

where the tune depression κ is the ratio of the betatron wavenumber with space charge to the betatron wavenumber without space charge,

$$F(x) = 4\frac{x+1}{x-1} - 2\left(\frac{x+1}{x-1}\right)^2 \ln(x) + 2\ln(x),$$

and ρ_0 is the ratio of the rms beam size at $t{=}0$ to the rms size of the matched beam.

In Figure 1, t_{cp} is plotted versus κ for various choices of ρ_0. We see that, for small tune depressions corresponding to space-charge-dominated beams, the oscillating-core/single-particle interaction ejects resonant particles within just a few core oscillations, but the time scale for this process grows rapidly as space charge weakens and the tune depression

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approaches unity. These analytic curves are consistent with results of numerical simulations of the interaction of single particles with a uniform, oscillating core [4]. The particles eventually go out of resonance with the core as their orbital amplitudes grow, and the maximum amplitude reached through this process is self-limiting.

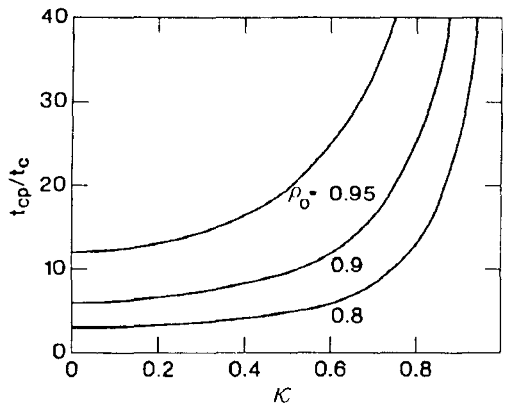

Fig. 1. Plots of t_{cp} vs. κ for various choices of rms mismatch of a uniform, oscillating core.

Fokker-Planck-Poisson Dynamics

The FPP formalism includes the effects of all modes on the particle dynamics and provides for a self-consistent calculation of the structure of the "core" via Poisson's equation. Thus, FPP includes both the global oscillatory mode considered in the previous section as well as all higher-order modes, and it requires no *a priori* assumption about the beam's structure. In the presence of a mode spectrum, a statistically few particles can resonantly interact with many modes and reach very large amplitudes in a few orbital periods. This contrasts with the self-limiting nature of the oscillating-core/single-particle interaction.

In the FPP formalism, the Fokker-Planck equation governs the evolution of the coarse-grained distribution and drives it toward Maxwell-Boltzmann equilibrium. The fluctuations generate dynamical friction and diffusion in velocity space which give rise to the Fokker-Planck "collision" term. Turbulence excited as a consequence of charge redistribution enhances these coefficients and converts free energy due to mismatch into thermal energy. Poisson's equation provides the coarse-grained space-charge force.

In general, the diffusion coefficient and relaxation rate may be expected to be functions of position, velocity, and time. For simplicity, and since we do not know these functions *a priori*, we ignore the position and velocity dependencies and model the beam as a fluctuating fluid in which particles execute Brownian motion. The diffusion coefficient is expressed as $D = \beta T/m$, in which β is the relaxation rate, m is the single-particle mass, and T is the "diffusive temperature" in energy units. We adopt a physically plausible phenomenological model of the diffusion coefficient by letting $T = T_\infty + (T_0 - T_\infty)\exp(-\beta_s t)$. Starting from temperature T_0, the beam strives to reach a Maxwell-Boltzmann distribution with temperature T_∞, and the heating occurs at the rate $\beta_s \geq \beta$ associated with "strong" turbulence.

To solve the coupled Fokker-Planck and Poisson equations self-consistently, we decompose the coarse-grained distribution function into complete sets of orthogonal polynomials:

$$\bar{W} = \sum_{m,n,q=0}^{\infty} \sum_{p=-\infty}^{+\infty} A_{m,n}^{p,q}(t)\psi_m(u_r)\psi_n(u_\theta)\phi_p^q(r)e^{ip\theta},$$

where the ψ's are Gauss-Hermite functions and the ϕ's are Gauss-Laguerre functions. Here and in what follows, we use cylindrical coordinates and the notation of Ref. [3]. The decomposition results in an infinite set of first-order, nonlinear differential equations for the expansion coefficients $A_{m,n}^{p,q}$ which is fully equivalent to the Fokker-Planck-Poisson equations. Solving this set of equations results in a self-consistent expression for the distribution function which may be used to calculate any desired moment as a function of time, including the particle-density profile and rms quantities.

We have written a FORTRAN code to solve for the expansion coefficients of a cylindrically symmetric beam (found from eq. (5.11) of Ref. [3] with $p=0$ and $\hat{a}=\hat{\alpha}=T_0/T$). The code has been benchmarked against a closed-form, analytic solution of the Fokker-Planck equation in which the orbits are all modeled as harmonic oscillators. It has also been verified to provide the correct final distribution function corresponding to thermodynamic equilibrium and for which the density profile can be calculated numerically directly from Poisson's equation. The solution process involves truncating the series of equations, solving the truncated series, then increasing the number of equations and solving the bigger series. If the solutions substantially agree, then one knows that a sufficient number of terms has been retained in the truncation. We found that a disadvantage of this method is an eventual sharp blowup of the solution triggered by a cascade to the highest-order expansion coefficient in the truncation. The numerical blowup will occur early in the solution if the beam is badly mismatched. However, we found that calculations for a modestly mismatched beam, like that expected to arise in a real machine, gave reasonably accurate results without early blowup using a relatively small number of expansion coefficients, although the accuracy of the halo's structure degrades with larger r. In the example below, 120 coefficients were found to provide sufficient accuracy: 3 values of m, 2 of n, and 20 of q.

The example we present here is representative of Axy-class linacs. We consider a linear focusing channel into which a beam with a Gaussian particle-density profile and Maxwellian velocity distribution is injected. The rms radius of the Gaussian beam is 1.2, where the unit of length is $(2T_0/m\omega^2)^{1/2}$ and ω is the angular betatron frequency without space charge. We take $\omega = \omega_0 = 1$ and $\omega_p = 1.5$. The rms radius of the equilibrium (matched) beam is 1.46. This example therefore resembles a transition to weaker focusing.

The input beam is modestly (20%) rms-mismatched, and it is believed that existing techniques for accelerator design would keep mismatches at transitions to about this level [5]. The ratio of the average Debye length [6] of the matched beam to its rms radius is 0.47, which indicates explicitly that space-charge forces are important to the beam dynamics, and therefore the beam is also mismatched in shape. The ratio of final-to-initial temperature is calculated from Reiser's theory [7] to be $T_\infty/T_0=1.21$. Because the mismatch is modest, we assume the turbulence is always weak and take $\beta_s=\beta$. Furthermore, we let $\beta=0.05\omega$, a relaxation rate compatible with results of numerical simulations. The initial and final density profiles are illustrated in Figure 2.

Evolution of the rms radius and rms emittance normalized to their initial values is shown in Figure 3. For comparison, curves calculated analytically by modeling all the orbits as harmonic oscillators are also shown. Space charge clearly has a significant quantitative effect on the evolution. It tends in this example to push particles out, increasing the size of the core both in configuration space and in phase space.

Evolution of the "halo" is illustrated in Figure 4. This figure shows the number of particles lying outside fixed radii equal to 1, 1.5, and 2. Curves calculated analytically with model harmonic-oscillator orbits are also shown, and once again it is seen that the nonlinear space-charge forces tend to push particles farther away from the beam axis, as is consistent with Figure 3. It is also seen that, in this example, the process is somewhat more prominent in the earlier stages of evolution than in the later stages.

In summary, we have applied a semianalytic formalism for calculating emittance growth and halo formation in high-current beams for which space charge is important. Unlike oscillating-core/single-particle models, this formalism strives for self-consistency. It also leads to relatively fast computation of the transient dynamics compared to N-body simulations. However, it incorporates an oversimplified model of the very complicated microscopic dynamics which involves at least one free parameter representing the relaxation rate. Thus, while the formalism accounts for all halo-formation mechanisms, its predictive accuracy is limited, particularly with regard to the detailed halo structure. Accordingly, future work should focus on improvements with the ultimate goal of including a self-consistent calculation of the Fokker-Planck coefficients.

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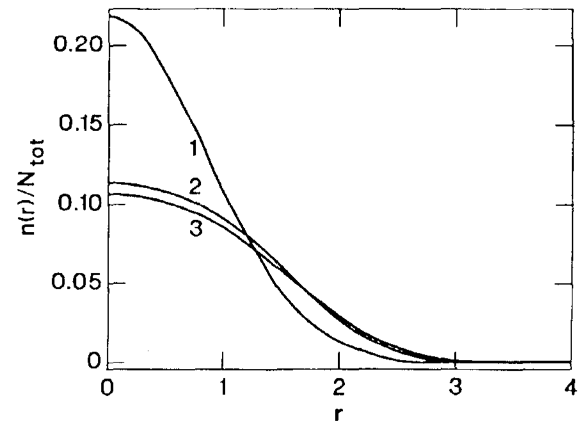

Fig. 2. Number-density profiles for initial mismatched beam (1), matched unheated beam (2), final $t\to\infty$ heated beam (3).

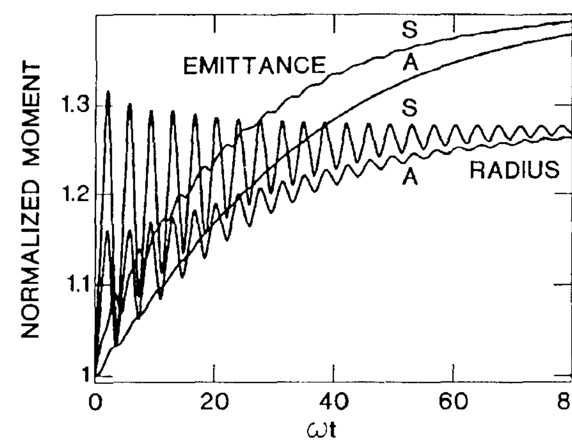

Fig. 3. Rms radius, emittance normalized to $t=0$. Self-consistent solutions are denoted "S". Analytic solutions with harmonic-oscillator orbits are denoted "A".

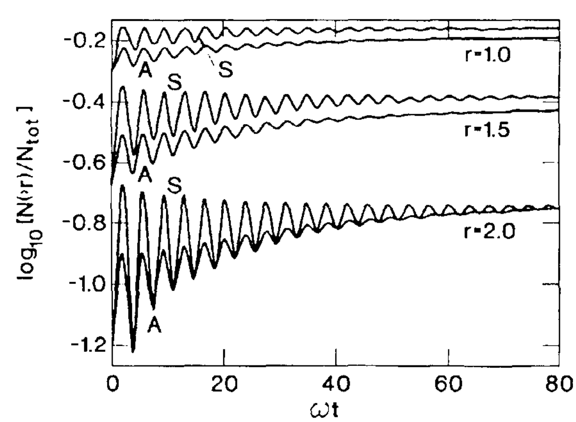

Fig. 4. Fractional number of particles outside $r=1$ (top), $r=1.5$ (center), and $r=2$ (bottom). "S": self-consistent solutions; "A": analytic solutions.