CENTRIFUGAL SPACE-CHARGE FORCE FOR BUNCHED BEAMS*

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Abstract

In 1990, E. Lee showed for a DC beam that the effect of the centrifugal space-charge force (CSCF) on an electron's transverse motion in a bend is cancelled to lowest order by the effect of the potential depression of the beam. However, in general this cancellation does not occur when the beam is bunched. We solve for the harmonic scalar and vector potentials and explicitly find the energy-independent transverse and longitudinal space-charge forces. We make estimates of the emittance growth as a function of bend angle and path length resulting from these forces.

Introduction

There is a trend in electron accelerators towards higher peak currents and lower transverse emittances for applications as diverse as linear colliders and free-electron lasers. Because the transverse space-charge forces are expected to scale inversely with the square of the relativistic mass factor these designs attain high peak current with magnetic bunch compression at relatively high energy.

Talman [1] and later Piwinski [2] described a space-charge force for long beams, known as the centrifugal space-charge force, that does not exhibit the relativistic cancellation if the beam path is circular. At first it was thought that this force would lead to energy-independent transverse motion and thus an energy-independent emittance growth. However, Lee [3] showed that in fact terms associated with the energy depression of a particle within the beam cancel this effect to first order in the beam radius divided by the bend radius.

We will discuss additional force terms for a bunch within a bend, related to Talman's and Piwinski's effect, but which arise when the beam current periodicity (or overall bunch length) is short in comparison to the beam pipe dimension and which are not cancelled by the effect of the beam's potential depression. We start with the wave equations and the Lorentz gauge condition and solve for the scalar and vector potentials using a perturbation analysis. We will then use the transverse equation of motion to find the emittance growth.

Perturbation Analysis for the Space-Charge Fields

In this section we will find explicit formulae for the potentials using a perturbation analysis. These potentials will then be used to derive the centrifugal-space-charge force as a function of bend radius and beam parameters in the next section. We will find an expression for the potentials for a ring of current. We can then use this expression as a Green's function for a uniform-density beam with circular cross section.

We assume a short bunch of current at a radius R, Fig. 1. This bunch can be harmonically decomposed into rings with a $e^{in(\theta-\omega t)}$ dependence.

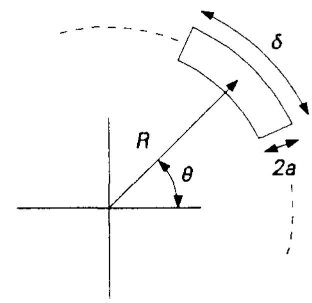

Fig. 1 Bunch of length δ and radius a in bend of radius R.

Let us now also assume that the vector potential for harmonic n is given by

$$\vec{A}_n = \left(\delta A_{r,n}, \frac{\beta}{c}\phi_n + \delta A_{\theta,n}, 0 \right) \qquad (1)$$

(A_z must remain zero because we have no current flow in the z direction). The wave equations for the scalar and vector potentials away from the ring of current lead to these expressions:

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\delta A_{r,n} + \left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}\right)\delta A_{r,n} + \frac{n^2 2x}{rR^2}\delta A_{r,n} - \frac{2in}{r^2}\left(\frac{\beta}{c}\phi_n + \delta A_{\theta,n}\right) - \frac{\delta A_{r,n}}{r^2} \qquad (2)$$

and

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\delta A_{\theta,n} + \left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}\right)\delta A_{\theta,n} + \frac{n^2 2x}{rR^2}\delta A_{\theta,n} + \frac{2in}{r^2}\delta A_{r,n} - \frac{\left(\frac{\beta}{c}\phi_n + \delta A_{\theta,n}\right)}{r^2}. \qquad (3)$$

The other important relation is the Lorentz gauge equation which leads to this additional expression:

$$\frac{\beta}{c}\phi_n\frac{x}{R} + \frac{i}{n}r\frac{\partial \delta A_{r,n}}{\partial r} + \frac{i}{n}\delta A_{r,n} - \delta A_{\theta,n} = 0. \qquad (4)$$

The scalar potential for bunch lengths of interest (> 0.1 ps) is very close to the scalar potential of the straight line beam [4],

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$$\phi_n = \frac{I_n}{2\pi\varepsilon\beta c}\ln\frac{\rho}{\rho_0} \quad , \tag{5}$$

where I_n is the harmonic current, $\rho^2 = x^2 + z^2$, and ρ_0 is the radius of the beam pipe (see Fig. 2).

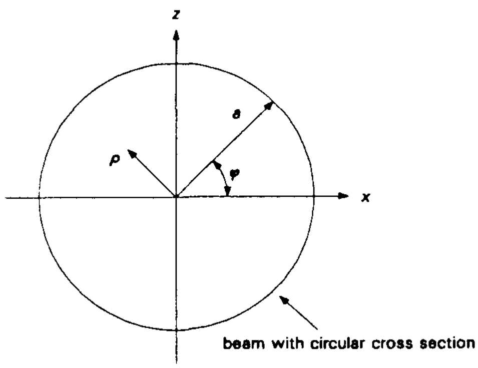

Fig. 2. Geometry used for the beam's local coordinate system. The beam is moving out of the plane of the page.

After keeping just the lowest order terms, these expressions can be integrated to yield in the low n regime

$$\delta A_{r,n} = \frac{in}{4Rr}\frac{I_n}{2\pi\varepsilon c^2}\left\{\rho^2\ln\frac{\rho^2}{\rho_0^2} - \rho^2 - 2z^2\right\} \tag{6}$$

and

$$\delta A_{\theta,n} = -\frac{1}{8Rr}\frac{I_n}{2\pi\varepsilon c^2}\left\{\rho^2\ln\frac{\rho^2}{\rho_0^2} - \rho^2 - 2z^2\right\}. \tag{7}$$

Transverse Equation of Motion

The transverse equation of motion resulting from the Lorentz force equation is

$$\begin{aligned}\ddot{x} &= \frac{v^2}{R}\left(-\frac{x}{R} + \frac{x^2}{2R^2} - \frac{x^3}{6R^3} + \cdots\right) \\ &\quad + \frac{e}{m\gamma}\frac{d\phi}{dr}\left(\beta_t^2 - \frac{1}{\gamma^2}\right) + \frac{e\beta^2}{m\gamma}G\end{aligned} \tag{8}$$

where

$$\begin{aligned}G &= \gamma_1\frac{mc^2}{R} + \frac{1}{e\beta^2}\sum_{n=1}^{n_{\max}}(\beta^2\frac{\phi_n}{r} \\ &\quad + v\frac{\partial\delta A_{\theta,n}}{\partial r} + \frac{v}{r}\delta A_{\theta,n} + \frac{invx}{rR}\delta A_{r,n})\end{aligned} \tag{9}$$

and $\gamma_1 mc^2$ is the deviation of the particle's energy from the nominal bend energy.

The first term in Equation (8) is not necessarily small, but does not lead to an emittance growth if the bend is achromatic. The second term on the right-hand side appears also for straight-line motion and is small, and we will ignore it. The G term is more complicated. If a particle's energy is not the nominal energy, but does not change, there will not be an emittance growth if the bend is achromatic. However, a change in a particle's energy can occur from work done on the particle in both the transverse and azimuthal directions. Assuming that the bunch does not deform appreciable along the bend, we can write the change in a particle's energy within a bend as:

$$\gamma_1 = \frac{eE_\theta}{mc^2}S + \frac{e}{mc^2}\int E_r dl \tag{10}$$

where the S is the distance through the bend that the bunch has travelled and the integral is from the center of the bunch to the particle's position. In this case, the integral can be rewritten as:

$$\int E_{r,n}dl = -\phi_n(x,z) - in\omega\int_{(0,0)}^{(x,z)}\delta A_{r,n}dl \ . \tag{11}$$

We can rewrite the Equation (8) (doing the harmonic summation where trivial) as:

$$\begin{aligned}\gamma\ddot{x} &= \frac{\beta^2 eE_\theta S}{mR} - \frac{\beta^2 e\phi(x,z,R\theta)}{mR} \\ &\quad - \sum_{n=1}^{n_{\max}}\frac{\beta^2 einv\int_{(0,0)}^{(x,z)}\delta A_{r,n}dl}{mR^2} - \frac{e}{m\gamma^2}\frac{d\phi}{dr} \\ &\quad + \beta^2\frac{e\phi(x,z,R\theta)}{mr} + \sum_{n=1}^{n_{\max}} e\frac{invx}{mrR}\delta A_{r,n} \quad .\end{aligned} \tag{12}$$

The harmonic summation is over all n until Equations (6) and (7) are no longer valid. As Lee found in the DC case, the second and fifth terms cancel exactly, and we are left with the usual energy-dependent term plus three noncancelling energy-independent terms.

Emittance Growth

The first emittance growth contribution will be from the net transverse space-charge force and the second will be from the effect of the electric field in the direction of motion of the bunch.

Harmonic summation of the $\delta A_{r,n}$ terms

First lets sum the radial vector potential terms over all harmonics. We have to be somewhat careful, because the vector potential solution is only valid for a line of charge. We can lump the transverse forces in a term $\hat{G}$, defined by

$$\gamma\ddot{x} = e\beta^2\frac{\hat{G}}{m} + \frac{\beta^2 eE_\theta S}{mR} \quad . \tag{13}$$

The summation over harmonics for a bunch of uniform current can be approximated well by integrating over n. We can then numerically integrate $\hat{G}$ over a full uniform, circular cross section and we find that the effective potential can be approximated by:

$$\hat{G}_{\text{integrated}} = 2.16 \frac{a^{3/2}}{R\sqrt{R\delta}} \frac{x}{a} \frac{I}{4\pi\varepsilon c} \ln\left(\frac{\rho_0}{a}\right) \quad . \tag{14}$$

This $\hat{G}$ is equivalent to an effective energy of a particle at that position within the bunch, and an rms spread in it will lead to a growth in the normalized emittance as the bunch travels a distance S along the bend:

$$\varepsilon_n = \frac{e\beta}{mc^2} S x_{rms} G_{rms} \quad . \tag{15}$$

If we assume the bunch has a parabolic current profile with peak current I_p, this leads to a normalized emittance growth of

$$\Delta\varepsilon_n = 0.54 \frac{a^{3/2}}{\sqrt{R\delta}} \ln\left(\frac{\rho_0}{a}\right) \frac{I_p}{I_A} a\alpha \tag{16}$$

where I_A is 17 kA.

Energy gain in the direction of motion

The second emittance growth mechanism results from the work done on the bunch by the electric field in the azimuthal direction, the direction of motion. The energy change of a particle at $(x,z,R\theta)$ within the bunch after a path length S is given by

$$\gamma_1 mc^2 = eE_\theta S. \tag{17}$$

In order to calculate the azimuthal field we start with the components from the individual harmonics:

$$E_{\theta,n} = -\frac{1}{r}\frac{\partial\phi_n}{\partial\theta} - \frac{\partial A_{\theta,n}}{\partial t} \quad . \tag{18}$$

After substituting in the values for the vector potential, we find

$$E_{\theta,n} = -\frac{in\phi_n}{r\gamma^2} + \beta^2 \frac{in\phi_n}{rR} x + in\omega\delta A_{\theta,n} \quad . \tag{19}$$

Because the longitudinal field only depends on E_θ, it is clear from writing the field this way that there does exist an energy-independent component to the longitudinal force. The first term is the same as we find for linear motion. The third term is of order $1/R$ higher than the second term (recall that $\omega = v/R$) and can be neglected.

Initially, the harmonic summation looks complicated; however we know that the straight-line motion terms must sum up to the usual longitudinal electric field. Thus, the energy-independent field for a line of current becomes

$$E_{\theta,l} = \frac{dI}{d\zeta} \frac{1}{2\pi\varepsilon\beta c} \frac{x}{R} \ln\frac{\rho}{\rho_0} \tag{20}$$

where $\zeta = R(\theta - \omega t)$, the position within the bunch. Now we can numerically integrate over all source positions to find the overall azimuthal field which leads to

$$E_\theta = 15.2 \frac{I_p \zeta}{4\pi c\varepsilon} \ln\left(\frac{\rho_0}{a}\right) \frac{x}{a} \frac{a}{R\delta^2} \quad . \tag{21}$$

The rms energy spread caused by this field after a drift S is

$$\Delta E_{rms} = 1.52 \frac{eI_p}{4\pi c\varepsilon} \ln\left(\frac{\rho_0}{a}\right) \frac{a}{R\delta} S \quad . \tag{22}$$

Integrating along the beam path within the bend, we find that the normalized emittance growth from the longitudinal space-charge force within the bend is

$$\Delta\varepsilon_n = 0.38\alpha^2 \ln\left(\frac{\rho_0}{a}\right) \frac{I_p}{I_A} a \frac{a}{\delta} \quad . \tag{23}$$

Conclusions

We have found that, contrary to common myth, the normalized emittance growth from nonlinear space-charge forces does not vanish as the beam is bent at higher and higher energies. There are two mechanisms for the emittance growth. First, there is an energy-independent transverse force. Also, there is an energy-independent longitudinal force that changes the particles' energy within the bend and breaks the bend's achromaticity.

Typical beam parameters for short-wavelength FELs and linear colliders are: beam radius of 100 μm, bend radius of 1 m, and bunch length of 1 ps (0.3 mm). The emittance growth from the transverse force is on the order of 0.1 μm per radian of bend angle and per kA of beam current. The emittance growth from the longitudinal force is per achromatic section of the bend. Using these parameters, there is about 2 μm of emittance growth times the bend angle (in radians) squared. This emittance growth can be reduced if each achromatic section bends the beam 0.1 radian or less.

Note that the emittance growth from both mechanisms is a strong function of the beam radius. It can quickly become significant if the beam is not focused strongly focused in the final bends where the beam is bunched.

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