REGENERATIVE BBU STARTING CURRENTS IN STANDING WAVE CAVITIES*

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Abstract

An analytical method for determining regenerative beam breakup (BBU) starting current, in which the contributions of single-cell field configuration and multi-cell structure mode are separated, is described. The field configuration within each cell is determined to close approximation through the use of mesh codes, which also relate the wall losses to the voltage drop along the beam path. The cell-to-cell amplitude variation may be determined by bead pull measurements on model cavities, or by assuming "idealized" structure modes. As an example, the $I_S Q_L$ product for TM_{110}-like modes of a 433-MHz, 5-cell, slot-coupled cavity is obtained.

Introduction

A conventional analysis of the regenerative beam breakup (BBU) instability in standing wave linac structures leads to the result

$$I_S Q_L = \frac{\omega\ U}{\overline{V}} \quad , \quad (1)$$

where I_S is the starting current, Q_L the loaded Q of the dipole mode, ω its angular frequency, and $\overline{V}$ is the potential drop along the electron path (averaged over all initial electron phases) when the dipole mode is excited to stored energy U. Near the axis, the relevant field components of the dipole mode may be represented as

$$B_x(x,y,z,t) = B_{Nx}(z)\ \cos(\omega t) \quad , \quad (2a)$$

and

$$E_z(x,y,z,t) = E_{Nz}(z)\ \frac{\omega\ y}{c}\ \sin(\omega t) \quad . \quad (2b)$$

Reasonable simplifying assumptions (beam is fully relativistic and does not change energy appreciably during transit of the structure) and choice of coordinates (origin at center of structure, which extends from z=-a to z=a) permit one to find $\overline{V}$ by first finding the path of the beam electrons under the influence of the fields, and then finding the work done on a charge moving along that path. For an even parity mode the result is

$$\overline{V}_+ = \frac{-e\ \omega}{2\gamma mc^3} \int_{-a}^{a} E_{Nz}(z)\ \cos\left[\frac{\omega z}{c}\right]\ dz$$

$$X \int_{-a}^{a} E_{Nz}(z)\ z\ \sin\left[\frac{\omega z}{c}\right]\ dz \quad , \quad (3)$$

where $\gamma = 1+E_k/E_o$ is the normalized total relativistic electron energy and the subscript "+" denotes the even parity case. (The result for odd parity modes is similar, with the sine and cosine functions interchanged and an overall change of sign.)

In the remainder of this paper, we show how the integrals can be separated into two factors, one depending on the detailed field configuration of the mode within a cell, and the other containing all the dependence on the structure eigenmode.

Evaluation of the Field Integrals

For the following discussion, we will assume that the structure has an odd number of cells designated by the index n, which runs from -k to +k. We will further assume that the cell-to-cell field strength pattern is one of the eigenmodes of an ideally tuned structure; i.e.,

$$E_{Nz}(z) = \sum_{n=-k}^{k} \left[J_{mn} E_{1z}(z-z_n) \right] \quad , \quad (4)$$

where $z_n = nc/2f_o$ is the axial offset of the nth cell (f_o being the fundamental linac frequency), and J_{mn} is the nth cell signed amplitude for the mth eigenmode of the N-cell ensemble

$$J_{mn} = \sqrt{\frac{2}{N}}\ \sin\left[\frac{nm\pi}{N} + \frac{m\ \pi}{2}\right] \quad (1 \leq m \leq N-1) \quad , \quad (5a)$$

and

$$J_{Nn} = \sqrt{\frac{1}{N}}(-1)^{n+k} \quad . \quad (5b)$$

Consider the first of the integrals in Eqn.(3), which we denote by the symbol I_1:

$$I_1 = \int_{-a}^{a} E_{Nz}(z)\ \cos\left[\frac{\omega z}{c}\right]\ dz$$

$$= \int_{-a}^{a} \left[\sum_{n=-k}^{k} \left[J_{mn} E_{1z}(z-z_n) \right] \right] \cos\left[\frac{\omega z}{c}\right] dz$$

$$= \sum_{n=-k}^{k} \left[J_{mn} \int_{-a}^{a} E_{1z}(z-z_n) \cos\left[\frac{\omega z}{c}\right] dz \right]$$

$$= \sum_{n=-k}^{k} J_{mn} \left[\begin{array}{l} \cos\left[\frac{\omega z_n}{c}\right] \int_{-a}^{a} E_{1z}(s)\cos\left[\frac{\omega s}{c}\right] ds \\ -\sin\left[\frac{\omega z_n}{c}\right] \int_{-a}^{a} E_{1z}(s)\sin\left[\frac{\omega s}{c}\right] ds \end{array} \right] , (6)$$

with $s = z - z_n$.

Assume that the full structure mode field has even parity. This could result from either an even parity single-cell mode (e.g., the TM_{110} mode) in combination with an even parity structure eigenmode (e.g., the pi mode of a 5-cell structure), or an odd parity single-cell mode (like the TM_{111} mode) in combination with an odd parity structure eigenmode (e.g., the 2p/5 mode of a 5-cell structure). For the sake of example, let us assume that we have an even parity single-cell mode. Then the second integral in Eqn.(6) vanishes, and

$$I_1 = I_c S_c \qquad , \quad (7)$$

with the definitions

$$I_c = \int_{-a}^{a} E_{1z}(s) \cos\left[\frac{\omega s}{c}\right] ds \quad , \quad (8)$$

and

$$S_c = \sum_{n=-k}^{k} \left[J_{mn} \cos\left[\frac{\omega z_n}{c}\right] \right] \quad . \quad (9)$$

Applying the same approach to the second integral in Eqn.(3) (which is complicated slightly by the presence of an additional factor of z in the integrand) we obtain

$$I_2 = I_c S_{zs} + I_{zs} S_c \qquad , \quad (10)$$

where

$$I_{zs} = \int_{-a}^{a} s\, E_{1z}(s) \cos\left[\frac{\omega s}{c}\right] ds \quad , \quad (11)$$

and

$$S_{zs} = \sum_{n=-k}^{k} \left[J_{mn}\, z_n \cos\left[\frac{\omega z_n}{c}\right] \right] . \quad (12)$$

Equation (3) can now be rewritten

$$\overline{V_+} = \frac{-e\omega}{2\gamma mc^3} I_c^2 \left[S_c S_{zs} + \frac{I_{zs}}{I_c} S_c^2 \right] . (13a)$$

Similarly, for odd parity modes,

$$\overline{V_-} = \frac{e\omega}{2\gamma mc^3} I_c^2 \left[S_s S_{zc} - \frac{I_{zs}}{I_c} S_s^2 \right] . (13b)$$

Evaluation of Single-Cell Integrals.

Figure 1 shows the URMEL output for the TM_{110} mode of a 433-MHz accelerator cavity with reentrant nose cones (Fig. 2). A number of numerical integrals are computed by the code, which integrates along a path which is offset a distance R0 from the axis, R0 being the beam hole radius (2.5 cm in the example). Thus, we can approximate our integral I_c by

$$I_c = \int_{-a}^{a} E_{1z}(s) \cos\left[\frac{\omega s}{c}\right] ds$$

$$\sim \frac{2\,c}{\omega R0} * INT.(EZ*COS(K*Z/B))DZ$$

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++++++++++++++,++++++++++++++++++++++++++++++++++++
++ RESULTS FOR THE ACTUAL CAVITY AS INPUT TO URMEL ++
++ ***** NOT ASSUMING ANY HALF CELL SYMMETRY ***** ++
++++++++++++++++++++++++++++++++++++++++++++++++++++
++ MODE TYPE                -  1-EM- 1           ++
++ FREQUENCY OF MODE        -  833.0360    MHZ   ++
++ FREQUENCY/CUT-OFF FREQU. -  0.2372923         ++
++ WAVE LENGTH OF MODE      -  0.3595341    M    ++
++ BEAM PIPE CUTOFF TE/TM   -  3514./7313.   MHZ  ++
++++++++++++++++++++++++++++++++++++++++++++++++++++
++ PARTICLE SPEED B(BETA)   -  1.000000         ++
++ INT.(   EZ        )DZ-  0.5093700      V      ++
++ INT.(   EZ*COS(K*Z/B))DZ-  0.2574406      V      ++
++ INT.(   EZ*SIN(K*Z/B))DZ-  0.3683289      V      ++
++ INT.( Z*EZ*COS(K*Z/B))DZ-  0.8916039E-02 V*M    ++
++ INT.( Z*EZ*SIN(K*Z/B))DZ-  0.2427933E-01 V*M    ++
++ INT.(Z*Z*EZ*COS(K*Z/B))DZ-  0.4243349E-03 V*M*M  ++
++ INT.(Z*Z*EZ*SIN(K*Z/B))DZ-  0.1709404E-02 V*M*M  ++
++++++++++++++++++++++++++++++++++++++++++++++++++++
++ TOTAL STORED FIELD ENERGY-  0.2442567E-11 VAS   ++
++ KLOSS (-V*V/4*ENERGY)   -  0.2066944E-01 V/PC  ++
++ VOLTAGE TAKEN AT    R0  -  0.2500000E-01 M     ++
++ SKIN DEPTH             -  0.2200579E-05 M     ++
++ Q WITH ALL END PLATES   -  38630            ++
++ Q WITHOUT LEFT  END PLATE-  50153            ++
++ Q WITHOUT RIGHT END PLATE-  38630            ++
++ Q WITHOUT BOTH END PLATES-  50153            ++
++ P WITH ALL END PLATES   -  0.3312633E-06 W   ++
++ P WITHOUT LEFT  END PLATE-  0.2200537E-06 W   ++
++ P WITHOUT RIGHT END PLATE-  0.3312633E-06 W   ++
++ P WITHOUT BOTH END PLATES-  0.2200537E-06 W   ++
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++ FULL CELL RESULTS  -  ASSUMING P-V*V/2/RS  AND ++
++ THAT THE INPUT IS THE RIGHT HALF OF A FULL CELL ++
++++++++++++++++++++++++++++++++++++++++++++++++++++
++ FREQUENCY              -  833.0360    MHZ   ++
++ FREQUENCY/CUT-OFF FREQU. -  0.2372923         ++
++ VOLTAGE TAKEN AT    R0  -  0.2500000E-01 M    ++
++ LOSS PARAMETER  AT R-R0 -  0.1356765E-01 V/PC  ++
++ SHUNT IMPEDANCE AT R-R0 -  0.3011982    MOHM  ++
++ (R/Q)           AT R-R0 -  5.179336     OHM   ++
++ (R/Q)'-(R/Q)/(K*R0)**2M  -  27.13404     OHM   ++
++ Q WITHOUT END PLATES    -  50153           ++
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Fig. 1. URMEL printout for TM_{110} mode of cavity of Fig. 2. Boxes are drawn around values used for the example.
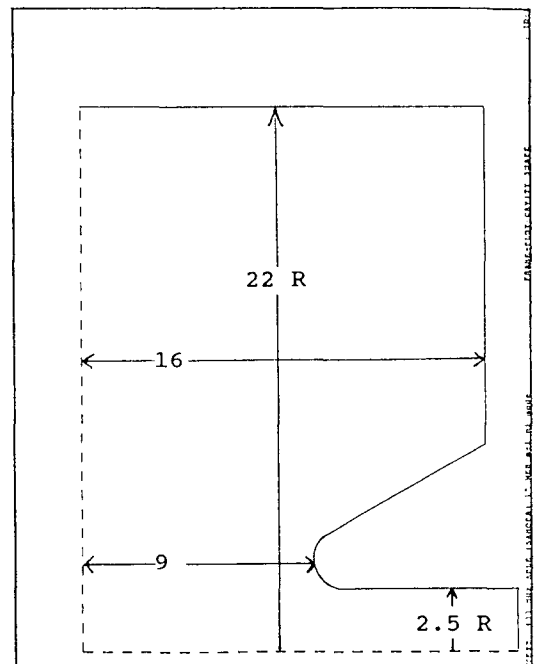

Fig. 2. Cell quadrant used in example calculation. Dimensions in cm.

$$= \frac{2 * 2.998*10^8 \text{m/s}}{2*3.14*8.35*10^8/\text{s} * .025 \text{ m}} * .2574 \text{ V}$$

$$= 1.18 \text{ V} \qquad\qquad , (14a)$$

and the integral I_{zs} by

$$I_{zs} = \int_{-a}^{a} E_{1z}(s) \; s \; \sin\left[\frac{\omega \; s}{c}\right] ds$$

$$\sim \frac{2 \; c}{\omega R0}*\text{INT.}(Z*EZ*\text{SIN}(K*Z/B))DZ$$

$$= \frac{2 * 2.998*10^8 \text{m/s}}{2*3.14*8.35*10^8/\text{s} * .025 \text{ m}}*.0243 \text{ V*m}$$

$$= .111 \text{ Vm} \qquad\qquad .(14b)$$

Evaluation of the Summations.

Note that for π-mode acccelerator structures the cell length is half a free space wavelength of the accelerator mode; therefore, $z_n = nc/2f_o$, where f_o is the accelerator mode frequency, whence

$$\frac{\omega z_n}{c} = \frac{2\pi f_m}{c} \frac{n \; c}{2f_o} = n\pi \frac{f_m}{f_o}$$

where f_m is the dipole mode frequency. Use of this equality in the summation of Eqn.(9) gives

$$S_c = \sum_{n=-k}^{k} J_{mn} \cos\left[\frac{\omega z_n}{c}\right]$$

$$= \sum_{n=-k}^{k} J_{mn} \cos\left[n\pi \frac{f_m}{f_o}\right] \qquad , (15)$$

which is obviously periodic in f_m/f_o.

Equation (12) defines summation S_{zs}; it is also a periodic function of the ratio f_m/f_o.

$$S_{zs} = \sum_{n=-k}^{k} J_{mn} \; z_n \; \sin\left[\frac{\omega z_n}{c}\right]$$

$$= \frac{c}{2f_o} \sum_{n=-k}^{k} J_{mn} \; n \; \sin\left[n\pi \frac{f_m}{f_o}\right] .$$

The summation index runs from $-k$ to $+k$, with the terms for positive and negative values making equal contributions. The $n=0$ term vanishes; therefore, the summation can be obtained by doubling the sum obtained for positive values of n:

$$S_{zs} = \frac{c}{f_o} \sum_{n=1}^{k} J_{mn} n \; \sin\left[n\pi \frac{f_m}{f_o}\right] . (16)$$

For the odd parity structure modes, similar arguments lead to these results:

$$S_s = 2 \sum_{n=1}^{k} J_{mn} \; \sin\left[n\pi \frac{f_m}{f_o}\right] , (17)$$

and

$$S_{zc} = \frac{c}{f_o} \sum_{n=1}^{k} J_{mn} \; n \; \cos\left[n\pi \frac{f_m}{f_o}\right] . (18)$$

Example: $\underline{TM_{110}}$ Band.

The structure mode-dependent summations above apply to any dipole mode band; the factor in square brackets on the right of Eqns.(13) depends on the ratio I_{zs}/I_c, and is therefore mode specific. For the mode found in the URMEL example above, we know the ratio of the single cell integrals is .094 m, and the reduced frequency is about 1.9. The average potential drop is plotted for each of the five modes in the band in Fig 3, where we see for reduced frequency between 1.8 and 2.0 the greatest value for $\bar{V}$ belongs to the m=1 mode, for which the average potential drop is 12 μV/Υ at the field amplitude assumed by URMEL.

The stored energy in the structure mode is the same as the single-cell stored energy in the URMEL calculation, because the eigenmodes are normalized. The corresponding stored energy for this mode is twice the half-cell value printed by URMEL, or 4.88 pJ. Using these values in Eqn.(1), we find $I_sQ_L \sim$23,000 A for a 5-MeV beam.

Conclusion.

The approach outlined above allows one to scan the URMEL output for single cell dipole modes, noting those for which the shunt impedance is largest. For each of these, the reduced frequency and the ratio I_{zs}/I_c can readily be found. These allow one to identify the structure mode with the largest value of $\bar{V}$ and to estimate the probable value of I_sQ_L/g.
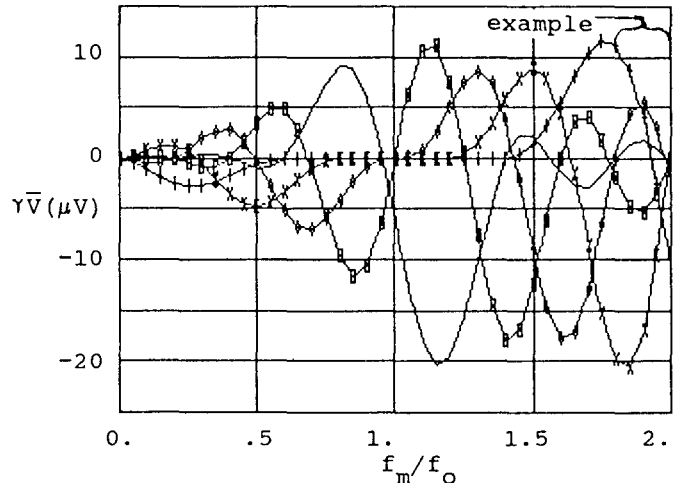
+:m=1 X:m=2 diam:m=3 sq:m=4 blnk:m=5

Fig. 3. Average potential drop plotted as a function of reduced frequency.