INVESTIGATION OF ALTERNATING-PHASE FOCUSING FOR SUPERCONDUCTING LINACS*

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Abstract

The paper describes a new model of alternating-phase focusing (APF) dynamics applicable to ion linacs with short independently controlled superconducting cavities. The equations of motion are derived for a cylindrically symmetric electric field represented by a traveling wave with continuous periodic phase modulation. Solutions are obtained and analyzed for both the linear and nonlinear particle motion. Problems of linear stability and overall longitudinal acceptance are solved using standard mathematical techniques for periodic systems; analytical results are obtained. It is shown that the main beam dynamical aspects of APF are adequately described by four parameters: equilibrium synchronous phase, phase modulation amplitude, length of APF period, and incremental energy gain. The model can be applied to study the feasibility of realizing APF in a low-β section of a proton linac.

Introduction

Previous works in the field [1, 2, 3, 4, 5] addressed APF in the context of a linac with a discrete number of accelerating gaps spaced in a predetermined manner to achieve a particular value of the synchronous phase in each gap (such as the case in the π-mode Wideroe linac and the Alvarez DTL). APF has also been studied for heavy ion linacs comprised of independently-phased short accelerating structures [6] where it was concluded that APF could be useful in some applications, namely low-velocity, low-mass particles, provided the rate of acceleration was sufficiently high. The application we have in mind is a proton linac with superconducting accelerating cavities of the type described in [7]. These low-β cavities are short, can be independently controlled in adjusting both the phase and the amplitude of the electric field, and were shown to produce very high accelerating gradients [8]. The purpose of the paper is to present a model which is thought to be a good description of essential APF physics for the superconducting linacs and a starting point in trying to determine the practical limits of APF.

Beam Dynamics in APF Linacs

Analytical Model and Assumptions

We assume that the electric field is described by a cylindrically symmetric traveling wave with a continuous phase modulation. Here, we choose the modulation to be sinusoidal:

$$E_z = E_0 \cos\left[\omega t - \int_0^z k(z')dz' + \phi_0 + \phi_1 \sin\left(\frac{2\pi z}{\Lambda}\right)\right], \quad (1)$$

where ω is the angular velocity and k is the wave number of the rf field, ϕ_0 is the equilibrium phase in the absence of APF, and Λ and ϕ_1 are the APF period and phase modulation amplitude respectively. For the central reference trajectory z_c, we choose

$$\omega t - \int_0^{z_c} k(z')dz' = 0. \quad (2)$$

In subsequent analysis, we will neglect the effect of the velocity change in one APF period; the reference particle is assumed to travel with a constant β and $k(z) = k = \frac{2\pi}{\beta\lambda}$.

We can compute the average accelerating gradient by integrating eq. 1 for the reference trajectory:

$$< E >= E_0 \cos\phi_0 J_0(\phi_1). \quad (3)$$

Equations of Motion

The equations of motion are

$$\frac{d^2 z}{dt^2} = \frac{q}{m} E_z(r, z; t), \qquad \frac{d^2 r}{dt^2} = \frac{q}{m} E_r(r, z; t), \quad (4)$$

with E_z given by eq. 1 and E_r determined to the first order in r by Maxwell's equations: $E_r(r, z; t) = -\frac{r}{2}\frac{\partial E_z}{\partial z}$. For an arbitrary longitudinal deviation from the reference trajectory $\Delta z = z - z_c$, the equation of motion becomes

$$\frac{d^2\Delta z}{dt^2} = \frac{qE_0}{m}\left\{\cos\left[\phi_0 - k\Delta z + \phi_1 \sin\left(2\pi\frac{z_c + \Delta z}{\Lambda}\right)\right] - \cos\left[\phi_0 + \phi_1 \sin\left(2\pi\frac{z_c}{\Lambda}\right)\right]\right\}. \quad (5)$$

We will first look at the APF linear motion.

Linear Stability

Let us define dimensionless parameters which we will use throughout this paper,

$$\Delta\phi \equiv -k\Delta z, \quad \tau \equiv \frac{z_c}{\Lambda}, \quad \nu \equiv \frac{\Lambda}{\beta\lambda}, \quad \eta \equiv \frac{qE_0\beta\lambda}{\frac{1}{2}m\beta^2 c^2}. \quad (6)$$

The linearized equations of motion are given by

$$\frac{d^2\Delta\phi}{d\tau^2} = \pi\eta\nu\left[(\nu - \phi_1\cos 2\pi\tau)\sin(\phi_0 + \phi_1 \sin 2\pi\tau)\right]\Delta\phi, \quad (7)$$

$$\frac{d^2 r}{d\tau^2} = -\frac{\pi}{2}\eta\nu\left[(\nu - \phi_1\cos 2\pi\tau)\sin(\phi_0 + \phi_1 \sin 2\pi\tau)\right] r. \quad (8)$$

By expanding the linear coefficients in eqs. 7, 8 in a Fourier series, we get the familiar Mathieu-Hill equations:

$$\frac{d^2\Delta\phi}{d\tau^2} - 2\left[B + \sum_{n=1}^{\infty} C_n \sin(2\pi n\tau + \theta_n)\right]\Delta\phi = 0, \quad (9)$$

*Work supported by the U. S. Army Strategic Defense Command under the auspices of the U. S. Department of Energy.

$$\frac{d^2 r}{d\tau^2} + \left[B + \sum_{n=1}^{\infty} C_n \sin\left(2\pi n\tau + \theta_n\right) \right] r = 0, \qquad (10)$$

where

$$B = \frac{\pi}{2}\eta\nu^2 J_0(\phi_1)\sin\phi_0, \qquad (11)$$

$$C_n = -\pi\eta\nu \left| J_n(\phi_1)\frac{\cos\phi_0}{\cos\theta_n} \right| \begin{cases} \nu & \text{if } n \text{ odd} \\ n & \text{if } n \text{ even} \end{cases}, \qquad (12)$$

$$\tan\theta_n = -\tan\phi_0 \begin{cases} n/\nu & \text{if } n \text{ odd} \\ \nu/n & \text{if } n \text{ even} \end{cases} \qquad (13)$$

The equations are analogous to those obtained in ref. [5] using a discrete thin-lens approximation and a standing wave approach. Here, however, the beam dynamics variables B and C_n depend only on four independent parameters: ϕ_0, ϕ_1, ν, and η; moreover, the dependence is given explicitly in an analytic form. Keeping only the $n = 1$ term, we obtain a well-known Mathieu equation for which we can compute stable region boundaries.

Fig. 1 (a) shows stability boundaries in the $\phi_1 - \nu$ space for $\phi_0 = 5°$, $\eta = 0.05$; fig. 1 (b) shows the effect of increasing the "acceleration parameter" η to 0.25.

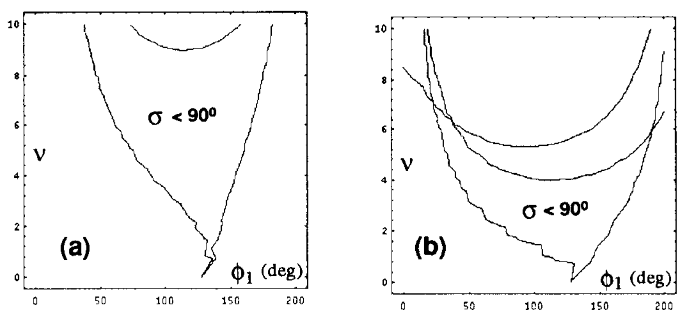

Figure 1: Stability boundaries for trajectories not exceeding 90° in either transverse or longitudinal phase advance with (a) $\phi_0 = 5°$ and $\eta = 0.05$, (b) $\phi_0 = 5°$ and $\eta = 0.25$.

We next turn to the nonlinear problem of calculating the longitudinal acceptance for the APF linac.

Longitudinal Acceptance

The equation of longitudinal motion is given by

$$\begin{aligned} \frac{d^2\Delta\phi}{d\tau^2} &= -\pi\eta\nu^2 \left\{ \cos\left[\phi_0 + \Delta\phi + \phi_1 \sin\left(2\pi\tau - \frac{\Delta\phi}{\nu}\right)\right] \right. \\ &\quad \left. - \cos\left[\phi_0 + \phi_1 \sin 2\pi\tau\right] \right\}. \qquad (14) \end{aligned}$$

We can calculate the effective potential for eq. 14 by using the averaging method given in ref. [9] and applied to the problem of longitudinal acceptance in ref. [5]. We do not review the method here.

We find the effective potential to be given by

$$U_{\text{eff}} = U_0 + \sum_{n=1}^{\infty} U_n \qquad (15)$$

with

$$U_0 = \pi\eta\nu^2 J_0(\phi_1)\left[\sin\left(\phi_0 + \Delta\phi\right) - \Delta\phi\cos\phi_0 - \sin\phi_0\right], \qquad (16)$$

$$U_n = \left(\frac{\eta\nu^2}{2}\right)^2 J_n^2(\phi_1)\frac{S_n}{n^2}, \qquad (17)$$

where

$$S_n = \begin{cases} s_1^2 + s_0^2 - 2s_1^2 s_0^2 \cos\left(n/\nu\Delta\phi\right) & \text{if } n \text{ odd} \\ c_1^2 + c_0^2 - 2c_1^2 c_0^2 \cos\left(n/\nu\Delta\phi\right) & \text{if } n \text{ even} \end{cases}, \qquad (18)$$

$$c_0 = \cos\phi_0, \quad c_1 = \cos\left(\phi_0 + \Delta\phi\right),$$
$$s_0 = \sin\phi_0, \quad s_1 = \sin\left(\phi_0 + \Delta\phi\right).$$

Given the effective potential, we can calculate the equation for the separatrix in the $(\Delta\phi, \frac{\Delta W}{W})$ space and the longitudinal acceptance. The separatrix is given by

$$\frac{\Delta W}{W} = \pm\frac{1}{\pi\nu}\sqrt{2\left[\Delta U - U_{\text{eff}}(\Delta\phi)\right]}, \qquad (19)$$

where

$$\Delta U = U_{\text{eff}}(\Delta\phi_c) \qquad (20)$$

and $\Delta\phi_c$ is the unstable fixed point of the motion.

Fig. 2 illustrates the relationship between the potential well $U_{\text{eff}}(\Delta\phi)$ and the stability boundaries.

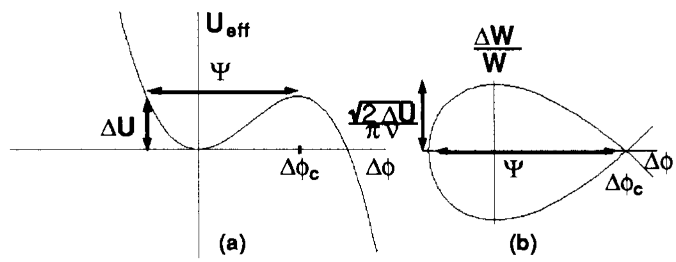

Figure 2: Relationship between (a) the effective potential $U_{\text{eff}}(\Delta\phi)$ and (b) the stable region in the $(\Delta\phi, \frac{\Delta W}{W})$ phase space.

The width of the separatrix Ψ is the distance between the values of $\Delta\phi$ at which $U_{\text{eff}}(\Delta\phi) = \Delta U$ (cf. fig. 2). The height of the separatrix is given by

$$\left(\frac{\Delta W}{W}\right)_{\max} = \frac{\sqrt{2\Delta U}}{\pi\nu}. \qquad (21)$$

The area of the stability region (acceptance) is

$$\alpha_L = 2\left(\frac{\Delta W}{W}\right)_{\max} \int_{\Delta\phi_c - \Psi}^{\Delta\phi_c} \sqrt{1 - \frac{U_{\text{eff}}}{\Delta U}}\, d(\Delta\phi). \qquad (22)$$

Below we give an explicit solution for α_L accurate to the second order in $\Delta\phi$.

Second-Order Solution. The effective potential U_{eff} given in eq. 15 can be expanded to $O\left(\Delta\phi^3\right)$ to yield

$$U_{\text{eff}}(\Delta\phi) = \frac{a}{2}\Delta\phi^2 - \frac{b}{3}\Delta\phi^3 + \cdots, \qquad (23)$$

where a is the square of the linear phase advance σ_L,

$$a = \sigma_L^2 = 2B + \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\left(\frac{C_n}{n}\right)^2 \qquad (24)$$

and b is given by

$$\begin{aligned} b &= \frac{\pi}{2}\eta\nu^2 J_0(\phi_1)\cos\phi_0 \\ &+ \frac{3}{8}\eta^2\nu^2\sin 2\phi_0 \sum_{n=1}^{\infty}(-1)^n J_n^2(\phi_1)\left(1 - \frac{\nu^2}{n^2}\right). \qquad (25) \end{aligned}$$

Then, the width of the separatrix Ψ and the acceptance α_L are calculated to be

$$\Psi = \frac{3}{2}\frac{a}{b}, \qquad \alpha_L = \frac{6}{5\pi\nu}\frac{a^{5/2}}{b^2}. \tag{26}$$

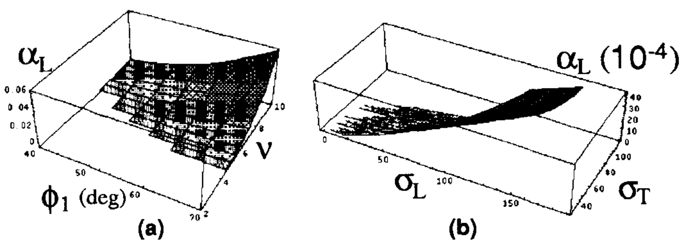

Figure 3: Plots of the longitudinal acceptance α_L for $\phi_0 = 5°$, $\eta = 0.1$. (a) Plot of α_L as a function of ϕ_1 and ν. (b) Plot of α_L as a function of σ_L and σ_T, the longitudinal and transverse phase advances respectively.

Fig. 3 shows the results of acceptance calculations for $\phi_0 = 5°$, $\eta = 0.1$ using eq. 26 and keeping only the $n = 1$ term in eqs. 24, 25. Computer simulations indicate that for most practical cases the second-order acceptance approximation is accurate with an error of less than 10%.

Amplitude Modulation

It has been suggested [1] that the effectiveness of APF can be increased by simultaneous modulation of the acceleration amplitude. Using the techniques presented above, we can incorporate the amplitude modulation into our model by replacing E_0 in eq. 1 by

$$E_0 \rightarrow E_0\left[1 + \epsilon\sin\left(\frac{2\pi z}{\Lambda} + \delta\right)\right]. \tag{27}$$

Here we choose a sinusoidal modulation with the same period Λ; the two new parameters are ϵ and δ, the strength and relative phase of the modulation respectively.

With the above change, it is a straightforward (though tedious) exercise to obtain the modified equations of motion and the effective potential.

The average accelerating gradient is now given by

$$< E > = < E >_0 \left[1 - \epsilon\cos\delta\tan\phi_0\frac{J_1(\phi_1)}{J_0(\phi_1)}\right], \tag{28}$$

where $< E >_0$ is the right hand side of eq. 3.

The linear equation of motion for $\Delta\phi$, keeping only the first term in the Fourier expansion, becomes

$$\begin{aligned}\frac{d^2\Delta\phi}{d\tau^2} = \quad & \pi\eta\nu^2\Delta\phi\Bigg\{\left[J_1(\phi_0)\sin\phi_0 + \epsilon\cos\delta J_1(\phi_1)\right] \\ & - 2\Bigg[1 + \epsilon\Bigg(\sin\phi_0\cos\delta J_1'(\phi_1) \\ & - \frac{\cos\phi_0\sin\delta}{\phi_1\nu}J_1(\phi_1)\Bigg)\Bigg]\sin 2\pi\tau \\ & - 2J_1(\phi_1)\Bigg[-\frac{\sin\phi_0}{\nu} + \epsilon\Bigg(\frac{\sin\phi_0\sin\delta}{\phi_1} \\ & + \frac{cos\phi_0\cos\delta}{\nu}\frac{J_1'(\phi_1)}{J_0(\phi_1)}\Bigg)\Bigg]\cos 2\pi\tau\Bigg\}.\end{aligned} \tag{29}$$

The constants in eqs. 9, 10 are modified as follows,

$$B \rightarrow B\left[1 + \epsilon\cos\delta\cot\phi_0\frac{J_1(\phi_1)}{J_0(\phi_1)}\right], \tag{30}$$

$$C_n \rightarrow C_n\sqrt{1 + \epsilon\xi_{1n} + \epsilon^2\xi_{2n}}, \tag{31}$$

where ξ_{1n}, ξ_{2n} are closed-form functions of δ, which we do not explicitly give here for lack of space.

Using the averaging method of ref. [9], we can recalculate the modified equations for the effective potential and obtain the new expressions for a and b in eqs. 23–26. The results of the changes on the longitudinal acceptance will be given elsewhere.

Conclusions

The model of the traveling wave with continuous phase modulation presented in this paper gives quantitative predictions to the problem of longitudinal stability in APF linacs. The model describes the physics of APF with four parameters and yields analytic solutions for the effective potential and the acceptance for the longitudinal motion. Modulation of the accelerating field amplitude can be straightforwardly incorporated into the model as well.

Future work on the model will focus on investigations of practical limits of APF in linacs with independent superconducting cavities, space-charge current limits, and ways to improve the acceptance by both the phase and the amplitude modulation.

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