CAVITY HEIGHT DEPENDENCE OF THE LONGITUDINAL IMPEDANCE FOR A PILLBOX CAVITY AT HIGH FREQUENCY*

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Abstract

The longitudinal coupling impedance for a cylindrically symmetric structure which consists of a pillbox connected to a beam pipe was previously studied by Gluckstern[4] using an integral equation method. The solution of the integral equation was found to be independent of the outer cavity radius b for $2k(b-a)^2/g \gg 1$, where the frequency is kc, the pipe radius is a, and the cavity length is g. In this paper, the requirement $2k(b-a)^2/g \gg 1$ is relaxed, yielding a more general expression for the impedance for a pillbox with dependence on $(b-a)$. In addition, it is shown that the analytic results agree well with the numerical data obtained using the scheme where the fields are matched in the transverse planes at the beginning and end of the pillbox.

Introduction

The longitudinal coupling impedance for a single pillbox connected to an infinite side tube has been extensively studied using various analytical methods[1, 2, 3]. In particular, Gluckstern[4] derived an integral equation for the axial electric field at the pipe radius, which is solved at high frequency by smoothing the kernels. All different approaches lead consistently to the $\omega^{-1/2}$ asymptotic high frequency behavior for the real part of impedance. Also, as a consequence of neglecting the contribution of the reflection from outer cavity wall to the wake field, the impedance expressions are found to be independent of b, which is the outer radius of the pillbox cavity. This approximation generally holds in the parameter region $2k(b-a)^2/g \gg 1$, where k is the frequency, a is the pipe radius, and g is the pillbox length. However, if $2k(b-a)^2/g \gg 1$ is not satisfied, one expects that the reflection of the diffracted wave from outer cavity wall will couple coherently with the traveling bunch beam which generates the wake fields, leading to the dependence of the impedance expression on the pillbox height $(b-a)$.

In this paper, we start with the cavity kernel in the integral equation given in Ref.[4], and obtain the smoothed expression for the cavity kernel with an explicit dependence on $(b-a)$. The integral equation is then solved using a Laplace transformation. It turns out that the impedance results thus obtained are in good agreement with the numerical data for various geometric parameters.

Review of Integral Equation Method

We first outline briefly the integral equation approach developed by Gluckstern. In Ref.[4], the impedance for an azimuthally symmetric cavity of general shape, which is connected to a beam pipe of radius a and has length g at $r=a$, is studied by calculating the axial electric field in the structure for the source current $J_z(r,z,t) = I_0\delta(x)\delta(y)e^{-jkz}$ with a metallic boundary condition ($k=\omega/c$, and the time dependent factor $e^{j\omega t}$ has been suppressed).

The integral equation for the axial electric field $f(z)(0<z<g)$ at the pipe radius is found to be

$$\int_0^g dz'[\hat{K}_p(z'-z)+\hat{K}_c(z',z)]G(z') = \frac{2\pi j}{a} \tag{1}$$

where $G(z) = 2\pi kaf(z)e^{jkz}/Z_0I_0$, and where $\hat{K}_p(z'-z)$ and $\hat{K}_c(z',z)$ are the modified pipe kernel and cavity kernel respectively. At high frequency when $ka^2 \gg g$, the expression for the smoothed part of the modified pipe kernels is

$$\langle\hat{K}_p(z'-z)\rangle \simeq \begin{cases} 0, & z'>z \\ \dfrac{j-1}{a}\left[\dfrac{\pi}{k(z-z')}\right]^{1/2}, & z'<z \end{cases} . \tag{2}$$

The explicit expression for the modified cavity kernel for a pillbox with outer radius b is given by

$$\hat{K}_c(z,z') = e^{-jk(z'-z)}\frac{8\pi}{g}\sum_{n=0}^{\infty}\frac{\cos n\pi\frac{z}{g}\cos n\pi\frac{z'}{g}}{1+\delta_{n0}} \times\sum_{m=0}^{\infty}\frac{P_1^2(\sigma_m a)[b^2P_1^2(\sigma_m b)-a^2P_1^2(\sigma_m a)]^{-1}}{k^2-\frac{n\pi}{g}^2-\sigma_m^2}, \tag{3}$$

where

$$P_0(\sigma_m r) = Y_0(\sigma_m r)J_0(\sigma_m b) - J_0(\sigma_m r)Y_0(\sigma_m b) \tag{4}$$

$$P_1(\sigma_m r) = Y_1(\sigma_m r)J_0(\sigma_m b) - J_1(\sigma_m r)Y_0(\sigma_m b) \tag{5}$$

with σ_m satisfying $P_0(\sigma_m a)=0$. At high frequency, the dominant modes can be approximated by

$$\sigma_m \simeq m\pi/(b-a), \tag{6}$$

and the asymptotic forms of the Bessel functions can be used. We note that the main contribution to the sums over n occurs near $n=kg/\pi$. The smoothed part of $\hat{K}_c(z',z)$ is then found to be

$$\langle\hat{K}_c(z',z)\rangle \simeq \frac{2\pi}{ag(b-a)} \times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\exp[j(n-kg/\pi)\theta]}{(1+\delta_{n0})\left[\frac{2k\pi}{g}\left(\frac{kg}{\pi}-n\right)-\left(\frac{m\pi}{b-a}\right)^2\right]} \tag{7}$$

*Supported by D.O.E. contract #DE-AC05-84ER40150

for $\theta \equiv \pi(z'-z)/g$. In the case when $2k(b-a)^2/g \gg 1$, the sums over n and m are approximated by integrals, which gives

$$\langle \hat{K}_p(z'-z)\rangle \simeq \begin{cases} 0, & z' > z \\ \frac{j-1}{a}\left[\frac{\pi}{k(z-z')}\right]^{1/2}, & z' < z \end{cases} . \tag{8}$$

The solution for the integral equation for the frequency region $ka^2 \gg g$ and $2k(b-a)^2/g \gg 1$ is then

$$G(z') \simeq \frac{1-j}{2\pi}\sqrt{\frac{\pi k}{z}}. \tag{9}$$

Therefore from the impedance formula

$$\frac{Z(k)}{Z_0} = \frac{1}{2\pi ka}\int_0^g dz\, G(z), \tag{10}$$

where $Z_0 = 120\pi$ is the impedance of free space, one gets the impedance for a pillbox cavity

$$\frac{Z(k)}{Z_0} = \frac{1-j}{2\pi a}\sqrt{\frac{g}{k\pi}}. \tag{11}$$

Dependence of Cavity Kernel on $(b-a)$ and the Impedance Result

In order to obtain an explicit expression for the impedance as a function of $(b-a)$, we here only calculate the sum over n in Eq.(7). Using the same scheme as applied in Ref.[4], we define $kg/\pi = N + \Delta$, where N is the nearest integer to kg/π and $-1/2 < \Delta < 1/2$. Thus we have $kg/\pi - n = \Delta - p$ for $p = n - N$. Since the major contribution to $\hat{K}_c$ occurs at $n \approx kg/\pi$, the sum over n can be extended to $-\infty$ and $(1+\delta_{n0})$ can be approximated by 1. Eq.(7) then becomes

$$\hat{K}_c(z,z') = \frac{\pi}{ag(b-a)} \times \sum_{m=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}\frac{\exp[-j(p-\Delta)\theta]}{\frac{2k\pi}{g}(\Delta-p)-(\frac{m\pi}{b-a})^2}. \tag{12}$$

The average of $\hat{K}_c(z,z')$ over the local frequency corresponds to

$$\langle \hat{K}_c(z,z')\rangle_k = \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{K}_c(z,z')\, d\Delta. \tag{13}$$

Therefore by writing $\omega = \Delta - p$, one gets

$$\langle \hat{K}_c(z,z')\rangle_k = \frac{1}{2ka(b-a)} \times \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega\, \frac{\exp[j\omega\pi(z'-z)/g]}{\omega - \frac{g}{2k\pi}(\frac{m\pi}{b-a})^2} \tag{14}$$

$$= \begin{cases} \frac{j\pi}{ka(b-a)}\sum_{m=-\infty}^{\infty} e^{\frac{j(z-z')}{2k}(\frac{m\pi}{b-a})^2}, & z > z' \\ 0, & z < z'. \end{cases} \tag{15}$$

Here the pole in Eq.(14) is chosen to be in the lower half of the complex ω plane so that the result will be consistent with that in Ref.[4] at the limit $\frac{2k(b-a)^2}{g} \gg \pi$.

With the modified cavity kernel expressed explicitly as a function of $(b-a)$ in Eq.(15), the integral equation (Eq.(1)) now becomes

$$\int_0^z dz' G(z')\frac{(j-1)\sqrt{\pi}}{a\sqrt{k(z-z')}} + \frac{\pi j}{ka(b-a)}\int_0^z dz' G(z') \times \sum_{m=-\infty}^{\infty} e^{\frac{j(z-z')}{2k}(\frac{m\pi}{b-a})^2} = \frac{2\pi j}{a}. \tag{16}$$

Here Eq.(2) is used for the modified pipe kernel assuming $ka^2 \gg g$. Let $F(s)$ be the Laplace transform of function $G(z)$. Applying the convolution theorem, we obtain the Laplace transform of Eq.(16),

$$\left[\frac{\alpha}{\sqrt{s}} + \beta\sum_{m=-\infty}^{\infty}\frac{1}{s-\sigma m^2}\right]F(s) = \frac{\gamma}{s}, \tag{17}$$

where the parameters are defined as

$$\alpha = \frac{(j-1)\sqrt{\pi}}{a\sqrt{k}}, \quad \beta = \frac{\pi j}{ka(b-a)},$$
$$\gamma = \frac{2\pi j}{a}, \quad \sigma = \frac{j\pi^2}{2k(b-a)^2}.$$

From Eq.(17), and using the identity

$$\sum_{m=-\infty}^{\infty}\frac{1}{\epsilon^2 - m^2} = \frac{\pi}{\epsilon\tan\pi\epsilon}, \tag{18}$$

we can solve for $F(s)$,

$$F(s) = \frac{\gamma}{\alpha\sqrt{s} + \beta s\sum_{m=-\infty}^{\infty}\frac{1}{s-\sigma m^2}} = \frac{(1-j)\sqrt{k}}{2\sqrt{s}}(1 - e^{2\pi j\sqrt{\frac{s}{\sigma}}}). \tag{19}$$

By properly defining the contour of integration, we can evaluate the function $G(z)$ from the inverse Laplace transformation:

$$G(z) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} ds e^{zs} F(s) = \frac{(1-j)\sqrt{k}}{2\sqrt{\pi z}}\left(1 - e^{\pi^2/\sigma z}\right). \tag{20}$$

The impedance is related to the function $G(z)$ by Eq.(10), which gives

$$\frac{Z(k)}{Z_0} = \frac{1-j}{4\pi a\sqrt{\pi k}}\int_0^g \frac{dz}{\sqrt{z}}\left(1 - e^{\pi^2/\sigma z}\right). \tag{21}$$

We then define $z = g/v^2$, and let v be the new integral variable in Eq.(21). Integrating the right-hand side of Eq.(21) by parts, we get

$$\frac{Z(k)}{Z_0} = \frac{1-j}{2\pi a}\sqrt{\frac{g}{\pi k}}\left[1 - e^{-j\xi} + 2j\xi\int_1^\infty dv\, e^{-j\xi v^2}\right]. \quad (22)$$

Here the parameter ξ is defined by

$$\xi = 2k(b-a)^2/g, \quad (23)$$

which characterizes the phase difference between the traveling bunch and the wave reflected by the outer wall of the cavity. Defining the correction function $f(\xi)$ as

$$f(\xi) = (1+j)\sqrt{\frac{\xi\pi}{2}} - e^{-j\xi} - 2\sqrt{\xi}\int_0^{\sqrt{\xi}} du\,(\sin u^2 + j\cos u^2), \quad (24)$$

which contains the Fresnel integrals, the expression for the impedance in Eq.(22) can be written in terms of the impedance in Eq.(11),

$$\frac{Z(k)}{Z_0} = \left[\frac{Z(k)}{Z_0}\right]_{\xi\to\infty}[1 + f(\xi)], \quad (25)$$

where

$$\left[\frac{Z(k)}{Z_0}\right]_{\xi\to\infty} \equiv \frac{1-j}{2\pi a}\sqrt{\frac{g}{\pi k}}. \quad (26)$$

For $\xi \gg 1$, $f(\xi)$ oscillates rapidly with reducing amplitude as ξ increases:

$$f(\xi) = e^{-j\xi}\sum_{l=1}^{\infty}\frac{j^l(2l-1)!!}{(2\xi)^l}. \quad (27)$$

As a consequence, the asymptotic behavior of the impedance at high frequency is consistent with the previous result:

$$\frac{Z(k)}{Z_0} = \left[\frac{Z(k)}{Z_0}\right]_{\xi\to\infty} \quad (28)$$

Comparison with Numerical Results

A scheme of numerical computation of the longitudinal coupling impedance for a chain of pillboxes has been developed in the ultrarelativistic limit[5]. In the process, a driving sinusoidal current is sent through the axis, generating scattered fields via the interaction with the environment. The scattered fields in each cavity and beam pipe section are expanded into TM modes. The solution for the fields is obtained by matching the fields on transverse planes, with outgoing wave boundary conditions applied on the far ends of the beam pipe. The numerical results thus obtained can then be compared with the analytic results presented in the previous section.

In Figures 1 and 2 we plotted the dependence of the real and imaginary parts of Z (in the unit of $Z_1 = 2\pi a\sqrt{g/k\pi}$) on the parameter $\xi = 2k(b-a)^2/g$, for $a = 1$, $g = \pi/4$, ka from 20 to 80, and $b = 1.1, 1.2$ and 1.5. Here the lengths are all in centimeters. The data for $(b-a) = 0.1, 0.2$ and 0.5 overlap in their common range of ξ. The figures show good agreement between the theory (Eq.(25)) and the numerical results, which display the behavior that Re(Z) and $-$Im(Z) oscillate arround Z_1 with decreasing amplitude as ξ increases.

It should be noted that the approximations in the above analysis do not apply well to the case with very small $(b-a)$. In Ref.[4], it is demonstrated that the major contribution to the sum over m in Eq.(3) comes from

$$\frac{m\pi}{b-a} \sim \left[\frac{k}{|z'-z|}\right]^{1/2} \gg 1. \quad (29)$$

When deriving Eq.(7) from Eq.(3), the asymptotic forms of the Bessel functions and σ_m are used under the assumption that the major contribution to the sum over m comes from terms with large m. However, in the case when $(b-a)$ is very small, the first several terms with lowest m also make important contributions to the sum. Therefore a better analysis requires a more accurate representation for the Bessel functions with lower m being applied to Eq.(3).

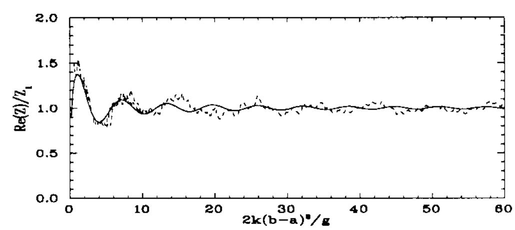

Figure 1: Analytical (solid curve) and numerical (dashed curve) results of Re(Z)/Z_1 vs. $2k(b-a)^2/g$ for $a = 1$ cm, $g = 1.5$ cm, $b = 1.1, 1.2$ and 1.5 cm, ka from 20 to 80.

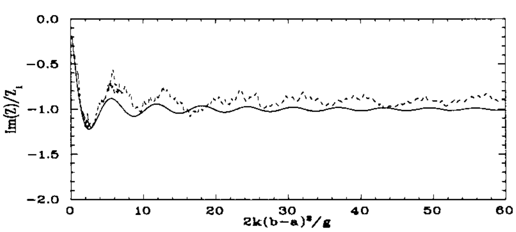

Figure 2: Analytical (solid curve) and numerical (dashed curve) results of Im(Z)/Z_1 vs. $2k(b-a)^2/g$ for $a = 1$ cm, $g = 1.5$ cm, $b = 1.1, 1.2$ and 1.5 cm, ka from 20 to 80.

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