LINAC BEAM DYNAMICS CALCULATIONS FOR LOW-CURRENT LARGE-EMITTANCE BEAMS*

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Abstract

The beam in PILAC, a superconducting linac for pions proposed at LAMPF, will have a larger momentum spread (7% dp/p) and occupy a larger transverse space (13 cm dia. bore) than is usual in high-beta linacs. To find the effects of this large phase space, a cavity element is being added to the MOTER code. With this addition, pions and other particles may be tracked through the injection line and the PILAC linac. In one option, the particles may be tracked cell by cell through a multicell cavity using formulas. The formulas are derived by integrating the energy gain and transverse impulse from the fields in a cell along the path of the particle. What is new in this analysis is that the transverse momentum is considered to be a significant part of the total momentum. The effect of a difference in velocity from the design velocity of the structure is considered. In another option still under development, field information is specified, and the particles may be tracked by stepwise integration.

Introduction

We will outline our derivation of an algorithm for tracking charged particles through linac structures when both the longitudinal and transverse emittances are large. First we will give a simplified analytical representation for the fields near the linac axis. Then we will give portions of the analysis of the longitudinal and transverse dynamics. Finally, we will compare three different methods for tracking particles using the MOTER [1] code, and outline areas needing further research.

Cavity Field Representation

In order to represent the fields near the axis of a multi-cell superconducting cavity, we find it convenient to represent the longitudinal electric field E_z as a sum of terms of the form

$$\cos\frac{n\pi z}{2h} ,$$

where n is a positive odd integer and h is half the cell length L. Thus E vanishes at the start and end of each cell. (This does not represent what happens at the entrance and exit of the cavity very well. We will come back to this point later.) In order to satisfy Maxwell's equations, a transverse magnetic field with $\cos \omega t$ time variation and $\cos(n\pi z/L)$ longitudinal variation has the form

$$E_z = \sum_n 2\, E_0 T_n\, I_0(k_{wn} r) \cos \omega t \cos\frac{n\pi z}{2h} , \tag{1}$$

$$E_r = \sum_n 2\, E_0 T_n\, \gamma_{wn}\, I_1(k_{wn} r) \cos \omega t \sin\frac{n\pi z}{2h} , \tag{2}$$

$$B_\phi = -\frac{1}{c}\sum_n 2\, E_0 T_n\, \beta_{wn}\gamma_{wn}\, I_1(k_{wn} r) \sin \omega t \cos\frac{n\pi z}{2h}, \tag{3}$$

where I_0 and I_1 are modified Bessel functions,

$$\beta_{wn} = \frac{v_d}{nc} , \tag{4}$$

$$\gamma_{wn} = (1 - \beta_{wn}^2)^{-0.5} , \tag{5}$$

and

$$k_{wn} = \frac{\omega}{\beta_{wn}\gamma_{wn} c} , \tag{6}$$

for cells designed to accelerate particles traveling at a velocity v_d, and where c is the velocity of light. (Note $v_d = 2\omega h/\pi$, in order that the particle travels two cell lengths in one rf period.) E_0 is the average field along the axis of the cell at its peak in time:

$$E_0 = \frac{1}{2h}\int_{-h}^{h} E_{z\ \mathrm{peak},\, r=0}\, dz . \tag{7}$$

The factors T_n are given by

$$T_n = \frac{\int_{-h}^{h} E_{z\ \mathrm{peak},\, r=0} \cos(\frac{n\pi z}{2h})\, dz}{\int_{-h}^{h} E_{z\ \mathrm{peak},\, r=0}\, dz} . \tag{8}$$

Note that T_1 is just the normal transit-time factor for a particle at the design velocity $v_z = v_d$.

For the cavities to be used for PILAC, the peak field along the axis is near to sinusoidal in z, and we find the values for T_n shown in Table 1.

TABLE 1
Transit-Time Factors for PILAC Cavity

n	T_n
1	0.7753
3	-0.0279
5	0.0013

Thus the series representing the fields in these cavities converges rapidly. At most, we need three terms.

Analysis of Beam Dynamics

We begin the derivation of the beam dynamics with the basic equations for momentum and energy change:

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$$\frac{d}{dt}(\mathbf{p}c) = qc(\mathbf{E} + \mathbf{v} \times \mathbf{B}) , \qquad (9)$$

and
$$\frac{dW}{dt} = q\,\mathbf{E} \cdot \mathbf{v} . \qquad (10)$$

These equations relate the momentum $\mathbf{p}c$ (MeV) and kinetic energy W (MeV) change per unit time t to the fields $\mathbf{E}$ and $c\mathbf{B}$ (MV/m) for a particle with velocity $\mathbf{v}$ (m/s) and relative charge q. (For a proton or a π^+, $q = +1$.) The relativistic quantities β and γ relate the particle velocity and mass to the speed of light and the rest mass: $v = \beta c$ and $m = \gamma m_o$.

Thus
$$W + W_o = \gamma W_o , \qquad (11)$$
and
$$pc = \beta\gamma W_o , \qquad (12)$$
where W_o is the rest energy (MeV) and $\beta c = v$.

Since
$$dt = \frac{1}{v_z}\,dz , \qquad (13)$$
we can write the integral form of the energy change equation (10) as

$$\Delta W = q\int_{-h}^{h} \left(\frac{v_x}{v_z}E_x + \frac{v_y}{v_z}E_y + E_z\right) dz . \qquad (14)$$

If the components of particle velocity v_x, v_y, and v_z change little in traversing one cell, then

$$\Delta W \approx q\frac{v_{xi}}{v_{zi}}\int_{-h}^{h} E_x\,dz + q\frac{v_{yi}}{v_{zi}}\int_{-h}^{h} E_y\,dz + q\int_{-h}^{h} E_z\,dz . \qquad (15)$$

We use subscripts i and f (initial and final) to indicate quantities at the start and end of a cell, and c, at the center of the cell. It is only the final term in Eq. (15) that is usually used in tracking protons and the like. We will retain all three terms.

We need to evaluate the components of E at the position of the particle and at the time that the particle is there. We assume v_z is nearly constant over the cell, and thus for time given by $t = z/v_z + \phi/\omega$, we have

$$\omega t \approx \frac{\omega z}{v_{zi}} + \phi_c . \qquad (16)$$

In order to find the correct rf phase to use for a given particle, we consider the given particle and another particle, a reference particle, which is moving along the axis at the design velocity for the cell. The time difference for arrival at the center of the cell is

$$t - t_d = t_d\left(\frac{v_d}{v_{zi}} - 1\right) , \qquad (17)$$

where $t_d = h/v_d$. We may express this in terms of the rf phase and $\beta_{w1} = v_d/c$:

$$\phi_c = \phi_i + \frac{\pi}{2}\left(\frac{\beta_{w1}}{\beta_{zi}} - 1\right) . \qquad (18)$$

We put our expressions in ωt in terms of $\frac{\pi z}{2h}$ for ease of integration:

$$\omega t = (1 + d)\frac{\pi z}{2h} + \phi_c , \qquad (19)$$

where
$$d \approx \frac{\beta_{w1}}{\beta_{zi}} - 1 . \qquad (20)$$

We also assume that r remains near its value at the center of the cell, r_c. The energy gain per unit charge in traversing a cell is then

$$\Delta W/q = \sum_n 2h\,E_0T_n\left[I_0(k_{wn}r_c)\,C_n(d) + \left(\frac{v_{xi}}{v_{zi}}\frac{x}{r} + \frac{v_{yi}}{v_{zi}}\frac{y}{r}\right)\gamma_{wn}\,I_1(k_{wn}r_c)\,S_n(d)\right] , \qquad (21)$$

where

$$C_n(d) = \frac{1}{h}\int_{-h}^{h}\cos\left[(1 + d)\frac{\pi z}{2h} + \phi_c\right]\cos\frac{n\pi z}{2h}\,dz , \qquad (22)$$

$$S_n(d) = \frac{1}{h}\int_{-h}^{h}\cos\left[(1 + d)\frac{\pi z}{2h} + \phi_c\right]\sin\frac{n\pi z}{2h}\,dz . \qquad (23)$$

The integrals in Eqs. (22) and (23) may be evaluated in closed form. If cosine and sine terms in d are approximated by the first two terms of their series representation, the results for n=1 are

$$C_1(d) = \frac{24 - \pi^2 d^2}{24 + 12d}\cos\phi_c , \qquad (24)$$

$$S_1(d) = -\frac{24 + 24d - \pi^2 d^2 - \pi^2 d^3}{24 + 12d}\sin\phi_c . \qquad (25)$$

For particles moving at the design velocity, d=0, these reduce to $C_1(0) = \cos\phi_c$, $S_1(0) = -\sin\phi_c$, as they should.

The new kinetic energy is then
$$W_f = W_i + \Delta W , \qquad (26)$$
so that
$$\gamma_f = (W_f + W_o)/W_o , \qquad (27)$$
and
$$\beta_f = \frac{1}{\gamma_f}\sqrt{(\gamma_f - 1)(\gamma_f + 1)} . \qquad (28)$$

We now wish to calculate the changes in transverse momentum p_xc and p_yc using Eq. (9). The force applied to the particle is proportional to $\mathbf{E} + \mathbf{v} \times \mathbf{B}$. The transverse part of this force is purely radial, since $E_\theta = 0$, and $\mathbf{v} \times \mathbf{B}$ is perpendicular to $\mathbf{B}$, which is only in the θ direction. Regardless of the direction of v,

$$(\mathbf{E} + \mathbf{v} \times \mathbf{B})_r = E_r - v_z B_\theta . \qquad (29)$$

Now
$$E_r - v_z B_\theta = \sum_n 2\,E_0T_n\,\gamma_{wn}\,I_1(k_{wn}r) \cdot \left(\cos\omega t\sin\frac{n\pi z}{2h} + \beta_z\beta_{wn}\sin\omega t\cos\frac{n\pi z}{2h}\right) . \qquad (30)$$

We assume that the change in transverse momentum is small, and we can then approximate the momentum change through the cell by applying an impulse at the center of the cell. From Eqs. (9), (30) and (19), we find

$$\Delta p_rc \approx \sum_n 2hq\,E_0T_n\,\gamma_{wn}\,I_1(k_{wn}r_c) \cdot D_n(d,\overline{\beta_z}) , \qquad (31)$$

with
$$D_n(d,\overline{\beta_z}) = \frac{1}{h}\int_{-h}^{h}\frac{1}{\beta_z}\cdot \left\{\cos\left[(1 + d)\frac{\pi z}{2h} + \phi\right]\sin\frac{n\pi z}{2h} + \right.$$

$$+ \overline{\beta_z}\beta_{wn} \sin [(1 + d)\frac{\pi z}{2h} + \phi] \cos\frac{n\pi z}{2h} \} dz . \quad (32)$$

In these expressions, $\overline{\beta_z}$ is the average value of β_z over the cell. The integral in Eq. (32) can also be found in closed form, provided that $\overline{\beta_z}$ is assumed to be a constant. Again approximating sine and cosine terms in d by the first two terms in their series representation, we find for n=1 that

$$D_1(d,\overline{\beta_z}) = -\frac{\sin \phi_c}{\overline{\beta_z}} \cdot [\frac{24 - \pi^2 d^2}{24}(1 - \overline{\beta_z}\beta_{wn}) +$$

$$+ \frac{48d + 24d^2 - 2\pi^2 d^3 - \pi^2 d^4}{96 + 96d + 24d^2}(1 + \overline{\beta_z}\beta_{wn})] . \quad (33)$$

Finally, since $x'_i = v_{xi} / v_{zi}$ and $p_{xi}c = \beta_{xi}\gamma_i W_o$,

$$p_{xf}c = x'_i \beta_{zi} \gamma_i W_o + \frac{x}{r}\Delta p_r c , \quad (34)$$

and

$$x'_f = \frac{p_{xf}c}{\beta_{zf} \gamma_f W_o} . \quad (35)$$

Corresponding expressions give the final momentum and divergence in y.

Tests in the MOTER Code

A cavity element is being added to the MOTER tracking and optimization code. The preliminary version of the cavity element has options for using any one of the following:

(a) Algorithm, assumes $\beta_z = \beta$, velocity close to design.
(b) New algorithm, $\beta_z \neq \beta$, velocity may differ from design.
(c) Field integration, simplified fields.

After some tests with the new algorithm, we found that the calculated trajectories differed significantly from the more exact trajectories found by field integration. We very recently extended the analysis to find the approximate offset in position at the end of the cell. We did this by integrating the transverse force up to an arbitrary point in the cell to estimate the change in transverse velocity at that point, and then integrating the velocity to find the transverse position offset. When offsets are introduced, care must be taken to ensure that the transformation from x_i , x'_i to x_f , x'_f is such that the determinant of the transformation matrix is $(\beta_{zi}\gamma_i)/(\beta_{zf}\gamma_f)$. A similar constraint must be imposed on the y coordinates. For the beams we have tested so far, β_z is close to β, and it is straightforward to do this.

We generated examples of results from these methods by tracking a set of particles (initially 360 MeV) through five cavities with seven cells each. The cavities were designed for 500 MeV particles. Some of the results for tracking an individual particle and for tracking particles in a K-V distribution are listed in Table 2. In method (b), using the new algorithm, only the n=1 terms in the field expansion were used. Method (b) did include the new analysis for finding transverse position offsets. For method (c), using field integration, n=1, 3, and 5 terms were used, but the results are very nearly the same if only n=1 terms are used. Using method (c) as our standard of comparison, we see that method (b) gives better values for all the parameters shown than method (a).

TABLE 2
Tracking Comparisons

	Method (a)	Method (b)	Method (c)
Individual particle parameters			
Energy, MeV	416.742	416.508	416.508
Position, x, mm	59.705	60.599	60.425
Angle, x′	0.004836	0.005461	0.005420
Position, y	-0.584	-0.617	-0.621
Angle, y′	-0.000277	-0.000282	-0.000283
Beam parameters			
Emittance, x	200.5306	200.624	200.639
α, x-plane	-1.7080	-1.9363	-1.9205
β, x-plane	18.568	19.162	19.060
Maximum x	61.020	62.003	61.840
Emittance, y	200.5782	200.725	200.753
α, y-plane	-0.5495	-0.6670	-0.6598
β, y-plane	9.312	9.544	9.477
Maximum y	43.218	43.769	43.618
Computer time, s	4.54	9.80	873.02

Discussion and Conclusions

The new algorithm developed for large-emittance beams is a definite improvement for both the longitudinal and transverse dynamics for particles whose velocities differ from the design velocity of the structure. In order to realize the improvement in transverse dynamics, we found it necessary to go beyond what would be expected from following the path for a simple impulse applied at the center of the cell, and to find a more accurate position offset in traversing a cell.

The simplified field expressions presented above may be used either to represent the mid-cell fields in superconducting cavities, or to serve as a test bed for evaluating different algorithms. We note that the fundamental component of these fields may be decomposed into a backward and a forward traveling wave. The forward wave is the main contributor to acceleration, and is in agreement with that presented by Hereward.[2] We plan to add the option for reading in more arbitrary fields, such as fields from the URMEL code, for use in MOTER. This would permit a more careful examination of the effects of end-cell fields, which decay exponentially into the adjoining beam tubes.

References

[1] H. A. Thiessen and M. M. Klein, "Spectrometer Design at LASL," in **Proc. of the 4th International Conf. on Magnet Technology, Brookhaven, 1972**, NTIS CONF-720908-12, pp. 8-17.

[2] H. G. Hereward, "The General Theory of Linear Accelerators," in **Linear Accelerators**, P. Lapostolle and A. Septier, eds., North-Holland, 1970, see pp. 21-22.