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COMPUTER DETERMINATION OF THE SCATTERING MATRIX PROPERTIES OF N-PORT CAVITIES*

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Abstract

We extend the Kroll-Yu (KY) method [1] of determining the resonance parameters of single-port cavities to multiport cavities by substituting the determinant of the S matrix for the reflection coefficient used in the KY method. In addition, we present a method for computing the elements of the S matrix in the two-port case, based upon the properties of cavity modes formed when the waveguides associated with the ports are shorted. Extension to the n-port case is discussed.

INTRODUCTION

This paper is concerned with the use of computer programs to determine the parameters of multiport microwave circuits. It is particularly addressed to the problem of determining Q_{ext} and the resonant frequency of waveguide loaded cavities, and to the problem of determining the elements of the S matrix. The MAFIA [2] and ARGUS [3] programs have a built in capability of computing the elements of the S matrix for any user selected frequency. This capability is based upon solving Maxwell's equation in the time domain, with appropriate incoming and outgoing wave boundary conditions imposed at terminating planes. These terminating planes are located in the waveguides connected to the ports of the structure at sufficient distance to damp out evanescent modes. The incoming wave is turned on smoothly at the selected frequency, and a sufficient number of time steps must elapse to allow the fields within the boundaries provided by the terminating planes to reach steady state. These programs, operating in a time independent mode, also have the capability of computing the frequencies and field configurations of the modes of the cavity formed by imposing electric or magnetic boundary conditions at terminating planes in the waveguides.

The KY method provides a procedure for determining the resonant frequency and Q_{ext} of the modes of a cavity with a single matched load waveguide output, or reduceable to a single output case by symmetry considerations. It is based upon the analytic properties of the reflection coefficient in the complex frequency plane. It requires values of the reflection coefficient at four real frequencies in the vicinity of the resonant frequency. In the context of this paper, it is convenient to think of the reflection coefficient as the (unique) component of the one-by-one S matrix suitable to the description of a one-port microwave circuit. Its values can be obtained from the above referenced programs operating in either the time-independent or the time-domain mode, but all practical applications of the method have made use of the time-independent mode. This paper is concerned with the extension of the KY procedure to multiport microwave circuits. In the first section, we show that the method applies unchanged if the reflection coefficient is replaced by the determinant of the S matrix. This recipe obviously applies to the one-port case as well, since the reflection coefficient is then the same as the determinant of the S matrix. In the second section, we address the problem of extending the time-independent method of determining the S matrix to the multiport case and present the progress we have achieved for the two-port case. It provides an alternative to the time-domain method and may be advantageous in certain circumstances. In addition, either can serve as a reliability check of the other.

ANALYTIC REPRESENTATION OF THE S MATRIX DETERMINANT

We begin by recalling the definition of the S matrix [4], assuming for simplicity of exposition that each waveguide propagates a single mode at frequencies of interest. Let $a_i \exp[jk_i(\omega)z_i]$ represent the amplitude coefficient for the incoming wave electric field in the ith waveguide normalized so that $\frac{1}{2}a_i a_i^*$ is equal to the power, and $b_i \exp[-jk_i(\omega)z_i]$ the corresponding quantity for the outgoing waves ($i = 1, ..., n$). Here z_i represents distance along the ith waveguide measured from its specified reference plane, chosen to increase in the outgoing direction. The reference planes are to be specified in a way which preserves any symmetry present in the circuit. The matrix elements S_{ij} of the S matrix are defined by the relation

$$b_i = S_{ij}a_j \ , \tag{1}$$

where all of the a's are taken to be zero except a_j. The S matrix defined in this way is symmetric, and for a lossless circuit (which we assume throughout), unitary for real values of ω. Thus $S_{ij} = S_{ji}$, and

$$S_{ik}S_{kj}^* = \delta_{ij} \text{ (summation convention assumed)} \ . \tag{2}$$

Following KY, we consider a solution of Maxwell's equations for complex eigenfrequency that satisfies an outgoing wave boundary condition for each propagating mode, and an evanescent wave boundary condition for all other modes. Writing this eigenvalue as $u + jv$, we identify u with the resonant frequency of the waveguide loaded cavity and $u/2v$ with the cavity Q_{ext}. We now consider the behavior of the matrix elements S_{ij} as their argument approaches the value $u + jv$. Because the eigenfrequency corresponds to a situation in which there are outgoing

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waves but no incoming waves, each S_{ij} must have a pole there. Thus in the vicinity of the eigenfrequency, we may write

$$S_{ij} = P_{ij} / (\omega - u - jv) + R_{ij} \ , \tag{3}$$

where we take the residue matrix P and the remainder matrix R to be constant near the pole. Since symmetry is an analytic property, it will be preserved in the analytic continuation to the complex plane so that both P and R are symmetric matrices. Equation (3) strongly suggests that the determinant of the S matrix $\det(S)$ is also singular at $\omega = u+jv$, perhaps even with a pole of order n. We shall show, however, that if the eigenmode is nondegenerate, then the pole in $\det(S)$ is of first order. Thus if the mode is nondegenerate, there is a unique field distribution associated with it apart from normalization. This means that the ratio of the outgoing wave mode amplitudes is independent of the way in which the mode is excited. In particular, if we imagine exciting the cavity by an incoming wave at the jth port at a frequency arbitrarily close to the eigenfrequency, the pole-dominated response must be such that the outgoing wave ratios are independent of the port from which the cavity is excited. Thus selecting a nonzero P_{kj} and defining V_i as the ratio of the outgoing wave amplitude from the ith port to that from the kth, we have

$$P_{ij} = V_i P_{kj} = V_i P_{jk} = V_i V_j P_{kk} \ . \tag{4}$$

The factorized form tells us that P is a matrix of rank one, and reference to the properties of matrices and determinants yields the result

$$\det(S) = \mathrm{Trace}\,(P/R)\det(R)/(\omega - u - jv) \ , \tag{5}$$

which exhibits the pole as first order. Because the S matrix is unitary for real values of the frequency, $\det(S)$ has absolute value one there and following KY we may represent $\det(S)$ by

$$\det(S) = -\frac{(\omega - u + jv)}{(\omega - u - jv)} \exp\left(-2j\chi(\omega)\right) = -\exp\left(2j\psi\right) \ , \tag{6}$$

where $\chi(\omega)$ is a real function analytic at $\omega = u + jv$. If one needs to deal with several poorly separated resonances one may replace the above by the form which exhibits the resonances of interest explicitly [1,5]. In applications of the KY method to one-port problems $\psi(\omega)$ has been obtained by computing the resonances of the cavity formed by shorting the waveguide at various lengths. It is given by $k(\omega)L$ where L is the distance between the short and the reference plane, and one obtains a (ψ, ω) pair from each mode for each length chosen. These pairs are then used to determine the resonance parameters. Such pairs could be used in exactly the same way for determing the parameters in the multiport case, but a method for obtaining them is required. As mentioned in the introduction, the entire S matrix can be determined for any specified frequency by means of the time domain solutions of Maxwell's equations, which would provide the needed quantities. In the next section, we discuss the use of time independent methods for this purpose.

DETERMINATION OF S MATRIX PARAMETERS

The two-port case

It is convenient to write the S matrix in the following manifestly unitary and symmetric form:

$$S_{11} = -\cos(\theta)\exp\left[j\left(\phi + d\phi\right)\right] \ , \tag{7}$$

$$S_{22} = -\cos(\theta)\exp\left[j\left(\phi - d\phi\right)\right] \ , \tag{8}$$

$$S_{12} = S_{21} = -j\sin(\theta)\exp\left(j\phi\right) \ . \tag{9}$$

We define unique values for the angles without limiting generality by requiring $-\pi/2 < \theta \leq \pi/2$, $-\pi < \phi \leq \pi$, and $-\pi/2 < d\phi \leq \pi/2$.

We form a cavity by shorting the waveguides at distances L_i from the reference planes and compute a set of resonant modes. The incoming and outgoing wave amplitudes in the two waveguides, evaluated at the reference planes, satisfy the following relations:

$$b_i/a_i = -\exp\left(j2\psi_i\right) \ , \tag{10}$$

$$a_2/a_1 = r\exp\left[j\left(\psi_1 - \psi_2\right)\right] \ . \tag{11}$$

Here ψ_i is $k_i L_i$ and r is the ratio of the incoming amplitude for the second port to that of the first port evaluated at the shorts. It is a real quantity with a sign which depends upon the relative sign conventions chosen for the fields in the two guides. It is readily computed from the transverse magnetic fields at the shorts, quantities which are available as computer printout. To relate r to these values it is necessary to respect the normalization conditions used in defining the amplitudes. If the two guides are identical and the fields are evaluated at corresponding points, then, up to a sign, r is equal to the field ratio. (Specification of the sign for one mode determines it for all.) We combine Eqs. (1) and (7) through (11), using algebra and trigonometry, to obtain

$$\tan(\theta) = 2\sin(d\psi)/(r - 1/r) \ , \tag{12}$$

$$\phi = 2\psi_1 - \bar{\phi} - d\phi \ , \tag{13}$$

where $d\psi$, D, and $\bar{\phi}$ are defined by

$$d\psi = \psi_1 - \psi_2 - d\phi \ , \tag{14}$$

$$D = \frac{r - 1/r}{|r - 1/r|} \sqrt{r^2 + 1/r^2 - 2\cos(d\psi)} \ , \tag{15}$$

$$\sin\bar{\phi} = r\sin(2d\psi)/D \ , \tag{16}$$

$$\cos\bar{\phi} = \left[r\cos(d\psi) - 1/r\right]/D \ . \tag{17}$$

Equations (12) through (17) determine θ and ϕ in terms of computer provided parameters and the still to be determined $d\phi$. In the case of a symmetric circuit—which is a case of great practical importance—$d\phi$ is zero, so that the S matrix is completely determined by the above, provided that one has chosen L_1 not equal to L_2. This means that from a single time-independent run, one can evaluate the S matrix at the frequency of each mode. For cases in which θ and ϕ vary slowly with frequency, one can— using interpolation—obtain the frequency dependence over an extended interval. We have found the method to be reliable and easy to use—but, for lack of space, applications will be reported elsewhere.

Since all the necessary equations related to a single choice of shorting lengths are satisfied with arbitrary values of $d\phi$, this quantity cannot be determined from a single run. In order to determine the S matrix of an unsymmetric two-port at a particular frequency, one requires two runs with different choice of shorting lengths, each of which yields a mode at that frequency, but with different absolute values of r. Once one has that much information, there are many ways to extract the S matrix parameters. Since it is difficult to choose two different length pairs that give a mode at exactly the same frequency, it is useful to find a procedure that allows use of interpolation to determine the needed properties of one of the modes. We have been using the following. We use the second pair of lengths only to determine $d\phi$, based on the readily derived relation

$$\tan(d\phi) = \frac{(r' - 1/r')\sin(D\psi) - (r - 1/r)\sin(D\psi')]}{[(r' - 1/r')\, cos(D\psi) - (r - 1/r)\cos(D\psi')]} \, . \tag{18}$$

Here, $D\psi = \psi_1 - \psi_2$ and the primed quantities refer to the second pair of lengths. If the waveguides have equal cutoffs, Eq. (18) can be simplified by taking $L'_1 = L'_2$, yielding

$$\tan(d\phi) = \frac{(r' - 1/r')\sin(D\psi)}{[(r' - 1/r')\cos(D\psi) - r + 1/r)]} \, . \tag{19}$$

It is expected that $D\psi'$ and r' can be found to sufficient accuracy by interpolation. We have applied this method to a step in height of a rectangular waveguide, and find good agreement with the analytic formulas in Marcuvitz [6].

The n-port case

While we have little experience with the $n > 2$ case, there is one method which appears to be practical for the case in which all of the waveguide cutoffs are equal. It is based upon recognition of the fact that each mode found with all ψ's equal corresponds to an eigenstate of the S matrix. Thus, with all ψ's equal, we have from Eqs. (1) and (10),

$$B_{i,k} = S_{ij} A_{j,k} = -\exp(j2\psi_{,k})\; A_{i,k} \, , \tag{20}$$

which is just the eigenvalue equation for the S matrix, with eigenvector $A_{i,k}$ and eigenvalue $-\exp(j2\psi_{,k})$. Here we have added the index k because there are n distinct eigenvalues and eigenvectors for each frequency, and we have used capital letters for the mode amplitudes to indicate that we choose to normalize the eigenvectors to unity (they form an orthonormal set). With this understanding, we have the following explicit expression for S_{ij}:

$$S_{ij} = -\sum_{k=1}^{n} A_{i,k}\; A_{j,k} \exp(j2\psi_{,k}) \, . \tag{21}$$

The determinant of S is simply the product of the eigenvalues, and the sum of the $\psi_{,k}$ replaces ψ in the KY formalism. For a given k, the ratios of the $A_{i,k}$ are determined from the transverse magnetic fields at the shorts in a manner analogous to the determination of r for the 2-port case. This approach is probably impractical if the cutoff frequencies are unequal because one does not know how to choose the L's to get equal ψ's. On the other hand, if all of the cutoff wavelengths are equal, one only has to choose all the L's equal. Then one obtains an S matrix eigenvalue and eigenvector at each frequency that appears in the cavity mode spectrum. To determine the S matrix at any particular frequency, one of course needs all n eigenvalues and eigenvectors. Starting with a particular choice of L and a particular mode, there are $n-1$ additional L values that will give linearly independent modes of the same frequency. The practicality of the method depends upon being able to determine them from computations at, say, $2n$ intelligently chosen lengths, combined with the application of interpolation to get adequate approximations to both the L values and the eigenvectors. If this program is successful, one series of computer runs would provide the S matrix over a broad frequency interval.

CONCLUDING REMARKS

The main results of this paper are the demonstration that the determinant of the S matrix plays the same role in the determination of resonance properties of multiport circuits as the reflection coefficient plays for one-port circuits, and the development of practical methods for determining the S matrix for two-port circuits using time-independent computational methods. Our experience in applying these methods is quite limited, but a number of practical applications are in progress.

One of us (NK) has profited from numerous discussions of the numerical aspects of this problem with Kwok Ko.

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