The Precise Measurement of Electric And Magnetic Fields in a Resonant Cavity

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Abstract

In the area of microwave measurement, perturbation method is widely used, but the perturbation formulas which are often used will not stay correct when the perturbing object approaches the wall of the resonant cavity, which is the result from the effect of the induced charges on the cavity wall when the object approaches it.

This paper takes this effect into account, and gives modified perturbation formulas which were proved by some experimental results. These modified perturbation formulas could be widely used in the measurement of the electromagnetic fields in linear accelerator and other microwave devices.

Introduction

To demonstrate the effect of the charges induced on the cavity wall when a perturbing object approaches it[1,2], it is reasonable to measure the field in a resonant cavity in which the field distribution is well known. Now let's measure the electric field along the axis of a pill box resonant cavity in TM_{010} mode by using a small metal ball. The uncorrected perturbation formula[3] is

$$(f_o^2 - f^2)/f_o^2 = (4/3)3\pi E_o^2 A^3 \qquad (1)$$

where f_o is the unperturbed resonant frequency of the cavity, f is the perturbed resonant frequency of the cavity, E_o is the magnitude of the electric field where the small ball is located and A is the radius of the ball.

It's known that the magnitude of the electric field along the axis of a pill box cavity in TM_{010} mode is a constant. The measured result, however, turns out to be not the same as we have expected. In practice, the nearer the ball is to the wall, the greater $f_o^2 - f^2$ will be, and the conclusion we get from eq.(1) will be that the field along the axis is not constant. It is obvious now that eq.(1) has to be modified in order to be used for precise measurement of field.

The reason why $f_o^2 - f^2$ become larger when the ball is near the wall is that charges are induced on the cavity wall when the perturbing object is introduced into the cavity, and the contribution of the induced charges to the field becomes appreciably large when the object is near the wall, so that the field measured by eq.(1) is the combination of the field to be measured and the field produced by the induced charges on the wall. What we should do now is to pick the original field out, and establish a straight forward relationship between the unperturbed fields and the two frequencies f_o and f.

It is well known that electric field could be perturbed by dielectric and metal materials, magnetic field, how-ever, could only be perturbed by metal material. In the following sections we will give directly the modified perturbation formulas to measure the electric, magnetic fields by using metal sphere perturbing objects. Experiments have been done to show the effectiveness of the modified perturbation formula. In the Appendixes the mathematic details have been given to show the way how these formulas are reached. As for the formula for dielectric purterbing sphere one could consult ref.[2].

*The work was done in Tsinghua University, Beijing and in IHEP.(Institute of High Energy Physics, Academia Sinica, Beijing).

Formula for Electric Field With Metal Sphere

According to perturbation theory[4] we have

$$f^2 = f_o^2(1 + \int_v (H_a^2 - E_a^2)dv) \qquad (2)$$

Starting from eq.(2) we distinguished two cases. The first of which will concentrate on the electric field and will be discussed in this section. Another will be discussed in the next section on the magnetic field and the combination of the two cases.

When there is no magnetic field where the sphere is located, eq.(2) could be changed into

$$(f_o^2 - f^2)/f_o^2 = \int_v E_a^2 dv \qquad (3)$$

After some mathematic derivations[Appendix 1], we get the modified perturbation formula

$$(f_o^2 - f^2)/f_o^2 = (4\pi/3)3E_o^2 A^3(1+\alpha) \qquad (E)$$

where E_o is the field to be measured without being perturbed by the ball, A is the radius of metal sphere and α is expressed as follows

$$\alpha = 4(A/C)^3 + 16(A/C)^6 + 55.6(A/C)^8 + 32(A/C)^9 \qquad (4)$$

where C is twice the distance from the centre of the ball to the wall. When the ball is far from the wall eq.(E) will return to eq.(1). Eq.(E) is well proved by the experiment, as shown in Fig.(1).

Formula for Magnetic field with Metal Sphere

When there is no electric field where the sphere is located, eq.(2) could be changed into

$$(f_o^2 - f^2)/f_o^2 = -\int_v H_a^2 dv \qquad (5)$$

After Appendix 2, we get the modified perturbation formula for magnetic field.

$$(f_o^2 - f^2)/f_o^2 = (4\pi/3)(3/2)A^3 H_o^2(1+\beta) \qquad (M)$$

Now we express E_r by letting n equal only 0,1,2. Therefore,

$$E_r = (2a_1^l A^{-3} + \sum_{i=1}^{l} \frac{2^i a_i^{l-1}(i+1)}{i!C^{i+2}} + E_o)P_1(cos\theta)$$

$$+(3a_2^l A^{-4} + \sum_{i=1}^{l} \frac{2^i a_i^{l-1}(2+i)!}{i!C^{3+i}} A)P_2(cos\theta) \qquad (A10)$$

The integral in eq.(3) should be performed in the following way

$$\int_v E_a^2 dv = \int_v E_r^2 dv = \int_0^A dA \int_v^{v+dv} E_r^2 dv \qquad (A11)$$

Now let $E_r = UP_1 + UP_2$ where

$$U = (2a_1^l A^{-3} + \sum_{i=1}^{l} \frac{2^1 a_i^{l-1}(2+i)!}{i!C^{2+i}} + E_o) \qquad (A12)$$

$$Q = (3a_2^l A^{-4} + \sum_{i=1}^{l} \frac{2^i a_i^{l-1}(2+i)!}{i!C^{3+i}} A) \qquad (A13)$$

$$\int_v E_a^2 dv = \int_o^A (\frac{4\pi}{3} U^2 A^2 dA + \frac{4\pi}{5} Q^2 A^2 dA) \qquad (A14)$$

From this equation we could get modified formulas for different order of precision by chosing l to be different numbers, for example, when we chose $l = 0$, we could get the unmodified formula expressed in eq.(1). When we chose l =2, after some tedious calculation we will get

$$(f_o^2 - f^2)/f_0^2 = \frac{4\pi}{3} 3E_o^2 A^3 (1+\alpha) \qquad (A15)$$

where

$$\alpha = 4(A/C)^3 + 16(A/C)^6 + 55.636(A/C)^8 + 32(A/C)^9 \qquad (A16)$$

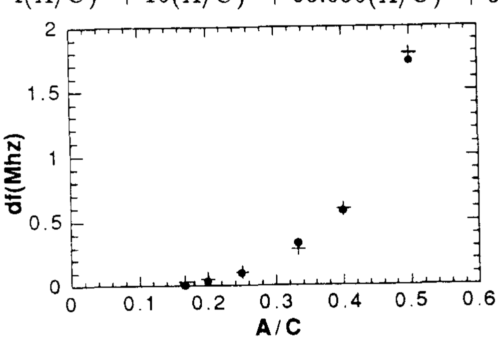

Fig.(1)

Appendix 2

In the same way as for the electric field, we consider the perturbation to the magnetic field by a small metallic ball. To start with we noted that Φ_m^l is the magnetic potential produced by the sphere introduced in the cavity.φ_m^{l-1} is the magnetic potential generated by the charges on the wall.Φ_{mi}^l is the magnetic potential inside the sphere by the magnetic field to be measured.

Therefore, to satisfy the boundary conditions on the surface of the sphere,that is R=A, and on the cavity wall $\Phi_m^{l-1} + \varphi_m^{l-1} = 0$ we have

$$\Phi_m^l - H_o R cos\theta + \varphi_m^{l-1} = \Phi_{mi}^l \qquad (A17)$$

$$\frac{\partial}{\partial R}(\Phi_m^l - H_o R cos\theta + \varphi_m^{l-1}) = 0 \qquad (A18)$$

where

$$\Phi_i^l = \sum_{n=0}^{\infty} b_n^l P_n(cos\theta) R^n \qquad (A19)$$

$$\Phi_{mi}^l = \sum_{n=0}^{\infty} a_n^l P_n(cos\theta) R^{-(n+1)} \qquad (A20)$$

$$\varphi_m^{l-1} = -\sum_{i=1}^{l} \frac{2^i a_i^{l-1}}{i!} \sum_{k=0}^{\infty} \frac{(k+i)!}{k!} P_k(cos\theta) \frac{R^k}{C^{k+1+i}} \qquad (A21)$$

From eqs.(A17) and (A18) we have

$$a_m^l = -\frac{A^{m+2}}{m+1}(H_o d_m 1 + \sum_{i=1}^{l} \frac{2^i a_i^{l-1}(m+i)! A^{m-1}}{i!(m-1)!C^{m+1+i}}) \qquad (A22)$$

When the sphere is far from the wall, Φ_m^l reduces to Φ_m^0[5], and

$$\Phi_m^0 = -H_o R cos\theta - \frac{H_o A^3 cos\theta}{2R^2} \qquad (A23)$$

By comparison between eqs.(A20) and (A23) we got

$$a_1^o = -\frac{H_o A^3}{2} \qquad (A24)$$

As for the magnetic field on the surface of the sphere, it only has component in θ direction,therefore we have

$$\int_v H_a^2 dv = \int_v H_\theta^2 dv \qquad (A25)$$

By asking approximation order $l = 1$ we got

$$H_\theta = -\frac{3}{2} H_o sin\theta + 3H_o sin\theta (A/C)^3$$

$$+(15cos^2\theta sin\theta - 3sin\theta)(A/C)^4 \frac{5}{2} H_o \qquad (A26)$$

$$\frac{(f_o^2 - f^2)}{f_o^2} = -\int_v H_\theta^2 dv = 2\pi \int_0^A \int_0^\pi H_\theta^2 A^2 sin\theta d\theta dA$$

$$= -\frac{4\pi}{3}\frac{3}{2} A^3 H_o^2 (1+\beta) \qquad (A26)$$

where

$$\beta = -2(A/C)^3 + \frac{4}{3}(A/C)^6 + 16.9(A/C)^8 \qquad (A27)$$

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$$\beta = -2(A/C)^3 + (4/3)(A/C)^6 + 16.9(A/C)^8 \quad (6)$$

When the ball is far from the wall, eq.(M) will return to unmodified one[3]

$$(f_o^2 - f^2)/f_o^2 = -(3/2)H_o^2(4\pi/3)A^3 \quad (7)$$

When there are both electric and magnetic fields at the same time where the ball is located, according to the theory expressed in eq.(2), combined perturbation formula is easily obtained

$$(f_o^2 - f^2)/f_o^2 = (4\pi/3)A^3 3(E_o^2(1+\alpha) - (1/2)H_o^2(1+\beta)) \quad (C)$$

When the ball is far away from the wall, eq.(C) will be reduced to unmodified perturbation formula[3] as expressed

$$(f_o^2 - f^2)/f_o^2 = (4\pi/3)A^3 3(E_o^2 - (1/2)H_o^2) \quad (8)$$

Till now we have given all the modified perturbation formulas of metal sphere.

Experiments

In the experiments we have used metal ball perturbing object to measure the electric field on the axis of the a pill box cavity. Before giving some experimental results we give some definitions to make the comparison between theory and experiment easier. We noted f_c to be the resonant frequency when the ball is in the center of the cavity. $\delta f_{th} = f_c - f_{th}(A/C)$ is the theoretical result, and $\delta f_{ex} = f_c - f_{ex}(A/C)$ is the experimental results. In the following Fig.(1) we will use only the notation δf but use + as theoretical value and · as experimental one. In Fig.(1) we used $A = 3mm$ copper ball to test Eq.(E). The result shows that theoretical modified perturbation formulas agree very well with the experimental results.

Conclusion

In this paper modified perturbation formulas concerning sphere metal perturbing object have been established and proved by the experiments. These formulas could be used in any kind of resonant cavity measurements.

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Appendix 1

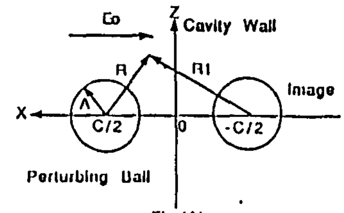

Fig.(A)

We treated this problem as a static one because the dimension of the perturbing object compared with the dimension of the wavelength of the microwave is very small. Therefore we have

$$\Phi_e^l = \sum_{n=0}^{\infty} a_n^l P_n(cos\theta) R^{-(n+1)} \quad (A1)$$

$$\varphi_e^{l-1} = -\sum_{i=1}^{\infty} \frac{2^i a_i^{l-1}}{i!} \sum_{k=0}^{\infty} \frac{(k+i)!}{k!} P_k(cos\theta) \frac{R^k}{C^{k+1+i}} \quad (A2)$$

where Φ_e^l is the potential produced by the sphere in the cavity. φ_e^{l-1} is the potential generated by the charges induced on the wall and l is the order of approximation. Both these potentials satisfy the boundary conditions on the wall

$$\Phi_e^{l-1} + \varphi_e^{l-1} = 0 \quad (A3)$$

When the sphere is far from the wall, Φ_e^l will be reduced to Φ_e^o [6].

$$\Phi_e^o = -E_o R cos\theta + \frac{E_o A^3}{R^2} cos\theta \quad (A4)$$

By comparing the two equations (A1) and (A4), we concluded that

$$a_1^o = A^3 E_o, a_n^o = 0, (n \neq 1), a_n^o = 0 \quad (A5)$$

To satisfy the boundary conditions on the surface of the ball, that is $R = A$ we get

$$\Phi_e^l + \varphi_e^{l-1} - E_o R cos\theta = constant \quad (A6)$$

$$\int\int_s \frac{\partial}{\partial R}(\Phi_e^l + \varphi_e^{l-1} - E_o R cos\theta) ds = 0 \quad (A7)$$

and consequently we get

$$a_m^l = E_o A^3 d_{ml} + \sum_{i=1}^{l} \frac{2^i a_i^{l-1}(m+i)! A^{2m+1}}{i! m! C^{m+1+i}} \quad (A8)$$

For the field on the surface of the ball in the direction of R, we have

$$E_r = \sum_{n=0}^{\infty}(n+1) a_n^l P_n(cos\theta) A^{-(n+2)}$$

$$+\sum_{i=1}^{l} \frac{2^i a_i^{l-1}}{i!} \sum_{k=0}^{\infty} \frac{(k+i)!}{k!} P_k(cos\theta) + E_o P_1(cos\theta) \quad (A9)$$