Abstract

At the present time the use of a static magnetic field has been the dominant method by which undulator synchrotron radiation has been produced. A microwave undulator utilizing a plane rectangular waveguide operating in the \(\text{TE}_{10n}\) mode presents a possible alternate means of producing this synchrotron radiation.\(^1\) Also, since a typical resonator may be many guide wavelengths in length it is possible to make the undulator source tunable over a broad spectral range.

Theory

Consider a rectangular waveguide operating in the \(\text{TE}_{10n}\) mode and having the dimensions shown in Figure 1. It will operate with field patterns as shown in Figure 2, where for guides operating with air or vacuum dielectric the guide wavelength \(\lambda_g\) is given by the relation:

\[
\lambda_g = \frac{\lambda_0}{\sqrt{1 - (\lambda_0/\lambda_c)^2}} \quad \text{(1)}
\]

\(\lambda_0\) is the free space wavelength and in general for a \(\text{TE}_{10n}\) mode we have the relationship

\[
\left(\frac{\lambda_0}{\lambda_c}\right)^2 = \frac{n^2}{2a^2} \quad \text{(2)}
\]

For the \(\text{TE}_{10n}\) mode we see that the cut-off wavelength, \(\lambda_c = 2a\) and hence

\[
\lambda_g = \frac{\lambda_0}{\sqrt{1 - (\lambda_0/2a)^2}} \quad \text{(3)}
\]

The electric and magnetic fields are given by the relations

\[
E_y = E_0 \sin \frac{2\pi}{\lambda_g} \sin wt \quad \text{(4)}
\]

\[
B_x = B_0 \cos \frac{2\pi}{\lambda_g} \cos wt \quad \text{(5)}
\]

where \(E_0\) and \(B_0\) are the peak electric and magnetic field strengths of the microwave field, respectively.

The equation of motion of a relativistic electron is given by

\[
\frac{dp}{dt} = -e(E + v \times B) \quad \text{(6)}
\]

substituting equations (4) and (5) in (6) gives the transverse motion of electrons in the microwave undulator as:

\[
\frac{d\delta y}{dt} = \frac{eE_0}{2m_0c^2} \left(1 + \frac{Z_y}{Z_w}\right) \cos \left(\omega t + \frac{2\pi}{\lambda_g}\right)
\]

\[
- \left(1 - \frac{Z_y}{Z_w}\right) \cos \left(\omega t - \frac{2\pi}{\lambda_g}\right) \quad \text{(7)}
\]

(Where \(B_y = \mu_0 H_y = \sqrt{\mu_0 / \epsilon_0} \omega E_0 / \lambda_y\) from Maxwell’s equations; the group velocity \(v_y = \lambda_y / \lambda_c\); \(Z_y\) the transverse wave impedance to is given by \(Z_w = (\lambda_y / \lambda_c) \epsilon_0 c^2\) and \(\beta_y\) is the electron velocity in the transverse direction divided by the velocity of light.) This shows that the electron is undulated by both a forward and backward wave component of the standing microwave fields.

Each of these waves will give rise to coherent radiation but we are interested in obtaining the short wavelength radiation contributed by the backward wave component.

Although the forward wave contribution to the undulator radiation spectrum may be made small by operating in a region where \(\lambda_c\) is very nearly equal to \(\lambda_0\), there will always be a component in the radiation spectrum resulting from the interaction of this wave component. Let us investigate the nature of this radiation. If we start with the equation of motion given by expression (7) we may compare this equation
with the equivalent expression for the conventional magnetic undulator which is given by
\[
\frac{d\theta}{dt} = \frac{eB}{m_0 \gamma} \cos \frac{2\pi}{\lambda_u}
\]  
(8)

Let us define an equivalent forward undulator field \( B_{uf} \) as
\[
B_{uf} = \frac{E_0}{2c} (1 - \frac{Z_0}{2u}) = \frac{E_0}{2c} \left( \frac{\gamma - \lambda}{\lambda} \right)
\]  
(9)

This may be compared with the equivalent expression for the backward wave,
\[
B_{ub} = \frac{E_0}{2c} (1 + \frac{Z_0}{2u}) = \frac{E_0}{2c} \left( \frac{\gamma + \lambda}{\lambda} \right)
\]  
(10)

we also may define the equivalent forward and backward wave wavelengths as
\[
\lambda_{uf} = \frac{\lambda}{\lambda + \lambda_0} \text{ or } \lambda_{uf} = \frac{\lambda + \lambda_0}{\lambda - \lambda_0}
\]  
(11)

\[
\lambda_{ub} = \frac{\lambda_0}{\lambda - \lambda_0} \text{ or } \lambda_{ub} = \frac{\lambda + \lambda_0}{\lambda - \lambda_0}
\]  
(12)

For a conventional undulator we also define a factor;
\[
K = \frac{eB}{2\pi mc} = 0.093 \text{ } B_u (KG) \text{ } \lambda_u (cm)
\]  
(13)

Note that here \( K_{uf} = 0.093 \text{ } B_{uf} \text{ } \lambda_{uf} = \lambda u / 2c = 0.093 \text{ } B_{ub} \lambda_{ub} = K_{ub} \text{ i.e. the } K \text{ factor for both forward and backward wave components is the same. For a single pass device there will be components in the spectrum of wavelengths } \lambda_{uf} \text{ and } \lambda_{ub} \text{ of amplitudes defined by the ratio } (\lambda - \lambda_1) / (\lambda + \lambda_1). \text{ This will give a long wavelength noise superimposed as the desired short wavelength radiation spectrum. For an undulator in a storage ring this component could result in a net deflection of the electron beam which may ultimately destroy the beam.}

In order that these waves give no net deflection of the beam we require that \( m \lambda_0 \text{, backward} = \lambda_u \text{ forward where } m \text{ is a positive integer, i.e.,}
\[
\frac{m \lambda_0}{\lambda} = \frac{\lambda_0}{\lambda} \text{ or } \lambda_0 = \frac{m \lambda}{m + 1}
\]  
(14)

But \( \lambda = \frac{\lambda_0}{\sqrt{1 - (\lambda_0/\lambda)^2}} = \frac{\lambda_0}{\sqrt{1 - (\lambda_0/2a)^2}} \)
for the TE\(_{10}\) mode. So we require that
\[
\frac{m+1}{m-1} = \frac{1}{\sqrt{1 - (\lambda_0/2a)^2}} \text{ or } \lambda_0 = 2a \sqrt{1 - (\frac{m-1}{m+1})^2}
\]  
(15)

We may choose \( \lambda_0 \) so that equation (15) is satisfied and then, outside of the undulator region itself, there is no net deflection of the electron beam by either the forward or backward waves. There is, however, a contribution to the undulator output spectrum from the forward and backward waves. Note that \( K \) is the same for both forward and backward wave contributions so that the wavelength and power of the undulator radiation from the forward and backward waves depends only on the undulator wavelengths \( \lambda_{uf} \text{ and } \lambda_{ub} \text{ and equivalent undulator field strengths } B_{uf} \text{ and } B_{ub}. \text{ The allowable operating wavelengths and equivalent undulator wavelengths and magnetic field strengths are given in Table I for a waveguide of broad dimension(a) cm.}

**Table I. Possible Operating Wavelengths for a Standing Wave Undulator.**

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \lambda_0 (cm) )</th>
<th>( \lambda_{uf} (cm) )</th>
<th>( \lambda_{ub} (cm) )</th>
<th>( B_{uf} )</th>
<th>( B_{ub} )</th>
<th>( K )</th>
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<td>1</td>
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<td>2.4824</td>
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<td>0.1154</td>
<td>1.5000</td>
<td>1.2000</td>
<td>0.6400</td>
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<td>0.6250</td>
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</tr>
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</table>

In order to eliminate the effect of the forward wave we would need to operate with \( \lambda_0 \) very nearly equal to \( \lambda_0 = 2a \). Let us consider the effect of both forward and backward waves on the undulator spectrum. Returning to equation (7) we may integrate this equation of motion to obtain
\[
y = \left[ \frac{eB}{8\pi m c^2 \gamma} \cos \left( \frac{\lambda}{\lambda_0} + \frac{1}{\lambda} \right) \right] 2\pi
\]  
(16)

\[
dy = \left[ \frac{eB}{8\pi m c^2 \gamma} \left( \sin \left( \frac{\lambda}{\lambda_0} + \frac{1}{\lambda} \right) - \sin \left( \frac{1}{\lambda_0} - \frac{1}{\lambda} \right) \right) \right] 2\pi
\]  
(17)

\[
dz = \left[ \frac{eB}{4\pi m c^2 \gamma} \right] \sin \left( \frac{\lambda_0}{\lambda} \right)
\]  

From Equation (16) we see that the maximum amplitude of the backward and forward wave oscillations are given by:

\[ y_{\text{MAX}}^{\text{y}} = \frac{eE_0^2}{8\pi \gamma c^2} \left( \frac{\gamma^2 g}{\lambda_0} \right) \left( \frac{\lambda}{\lambda_0} \right) \]  
\[ y_{\text{MAX}}^{\text{y}} = \frac{eE_0^2}{8\pi \gamma c^2} \left( \frac{\gamma^2 g}{\lambda_0} \right) \left( \frac{\lambda}{\lambda_0} \right) \]  

(18)

(19)

We may also determine the maximum angular deflection of both forward and backward waves from Equation (17) as

\[ \frac{dy}{dz} \text{MAX} = \frac{eE_0^2}{4\pi \gamma c^2} = \frac{K}{\gamma} \]  

(20)

Expression (21) is exactly that obtained for the static magnetic field undulator in Reference (3). It is also possible to operate the undulator in a travelling wave mode and thus avoid the effects of the forward wave. Alternatively an external magnetic field could be applied in order to suppress the effect of the forward wave.

**Radiation Characteristics**

The characteristics of the radiation from the microwave undulator are exactly the same as those from a static magnetic undulator. According to Hofmann the wavelength and intensity of the undulator radiation are given by

\[ \lambda = \frac{\lambda_0}{2\gamma^2} \left( 1 + \frac{K^2}{2} \right) \]  

(21)

\[ d\lambda = \frac{e^2 \gamma^4 K^2}{\lambda_0^2} \left( 1 + \frac{K^2}{2} \right)^3 \]  

(22)

where \( \gamma^2 = 1 + \frac{K^2}{2} \)

The angles \( \theta \) and \( \phi \) are observation angles defined in Figure 3 and \( \epsilon_0 \) is the permittivity of a vacuum. Integrating equation (23) over \( \phi \) and substituting from (22) we get

\[ d\lambda = \frac{3P_o \gamma^2 K^2}{\lambda_0^2} \left( 1 - \frac{2\lambda_1}{\lambda} + \frac{2\lambda_2}{\lambda^2} \right) \]  

(24)

where \( P_o \) is the total radiated power from an electron given by

\[ P_o = \frac{1}{4\pi \epsilon_0} \frac{e \gamma^2 B^2}{3m_c^2 \left( 1 + \frac{K^2}{2} \right)^2} \]  

(25)

and \( \lambda_1 \) is the wavelength of the on-axis radiation, i.e.

\[ \lambda_1 = \frac{\lambda_0^2}{2\gamma^2} \left( 1 + \frac{K^2}{2} \right) \]  

(26)

Since the particles in the beam have a statistical longitudinal distribution of a length much greater than the wavelength of the radiation, the radiation is not coherent and we can add up the powers radiated by each particle. So for a current \( I \) and an undulator length \( d \) we obtain

\[ P_o = \frac{1}{4\pi \epsilon_0} \frac{e \gamma^2 B^2}{3m_c (1 + \frac{K^2}{2})^2} \]  

(27)

**Higher Order Modes**

For a resonator installed in a storage ring other possible resonant modes are of concern since the beam may excite them. For a plane guide of dimensions \( a,b,d \) as shown in Figure 1 there are many possible modes with resonant wavelength, \( \lambda_0 \), given by the relation

\[ \lambda_0 = \frac{2}{\sqrt{\left( \frac{a}{b} \right)^2 + \left( \frac{b}{d} \right)^2}} \]  

(28)

for either transverse electric \( TE_{mn} \) or transverse magnetic \( TM_{mn} \) modes. Those with the longest cut off wavelength are the \( TM_{11n} \) and \( TE_{11n} \) families of modes both of which have a longest cut-off wavelength, \( \lambda_c \) given by

\[ \lambda_c = \frac{2a}{\sqrt{1 + \left( \frac{b}{d} \right)^2}} \]  

(29)

For a standard guide with \( a/b = 2 \) this gives \( \lambda_c = 0.894a \) so that the useful wavelength region for operation of a resonator in the \( TE_{10n} \) mode is given by

\[ 0.894a < \lambda_0 < 2a. \]  

Note that for the \( TE_{20n} \) family the cut-off wavelength \( \lambda_c = a \), so it may be desirable to restrict the operating wavelength region to a \( \lambda_0 < 2a. \) This clearly conflicts with the need to make \( \lambda_g \) very nearly equal to \( \lambda_0 \) or \( \lambda_0 << 2a. \)
Possible Undulator Configuration

Let us consider a standard waveguide of internal dimensions \( a \approx 2 \text{ cm}, \ b \approx 1 \text{ cm} \) and a length of 100 cm. The lowest practical operating wavelength, \( \lambda_0 \), is given by equation (15) which defines the cut-off wavelength for the TE\(_{11n}\) TM\(_{11n}\) mode families at 1.95 cm. Let us choose \( \lambda_0 = 2 \text{ cm} \) which is well below the cut-off wavelength of 4 cm for the TE\(_{100}\) mode.

The resulting undulator parameters are as follows:

- Operating Frequency, \( f_0 \): 15 GHz
- Undulator Length: 100 cm
- Guide Wavelength, \( \lambda_g \): 2.31 cm
- Equivalent Undulator Field, \( B_u \): 466 Gauss
- Undulator Quality Factor, \( K \): 0.05
- Undulator Wavelength, \( \lambda_n \): 1.07 cm
- Shunt Resistance, \( R/b \): 306 MS/m
- O Value: 14750
- Number of Undulations, \( n \): 87
- Peak Electric Field Strength, \( E_0 \): 15 MV/m
- Power: 367 kW

We are interested in attaining undulator wavelengths in the order of 0.5 cm to 1 cm so let us consider a waveguide half filled with a ceramic dielectric or a ferrite material as shown in Figure 4.

![Figure 4](image)

Figure 4. Cross-section of a dielectric loaded resonator of length 'd'.

The resonant wavelength for a fully filled waveguide of material of permeability \( u \) and permittivity \( e \) is given by:

\[
d = \frac{\lambda_0}{2} = \frac{\lambda_0}{\sqrt{\frac{\mu_0 e_0}{\mu e}}}
\]  

(30)

For a guide which is half dielectric and half vacuum we have

\[
d' = \frac{\lambda_0}{2} = \frac{\lambda_0}{\sqrt{\frac{\mu_0 e_0}{\mu e}}} + \frac{\lambda_0}{\sqrt{\frac{\mu_0 e_0}{\mu e}}}
\]  

(31)

For a 2 cm x 1 cm waveguide half filled with a dielectric which had a value of \( \sqrt{\frac{\mu e}{\mu_0 e_0}} = 3 \) operated at a wavelength of 2 cm we obtain an undulator wavelength \( \lambda_u = \lambda_0/2 = 0.746 \text{ cm} \).

Conclusion

The microwave undulator represents a viable option for undulator wavelength down to about 1 cm where peak voltage and available microwave power considerations limit its effectiveness. For undulator wavelengths in the 5 to 10 cm region a pulsed microwave undulator is an attractive proposition for either storage ring or linac, single pass, operation.

References


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