BEAM CAVITY INTERACTION

R. L. Gluckstern*
Fermi National Accelerator Laboratory, Batavia, IL 60510
R. K. Cooper
Los Alamos National Laboratory, Los Alamos, NM 87545

Introduction

In rf accelerating cavities, the fundamental excitation serves to accelerate a beam which is bunched in synchronism with some harmonic of this excitation. As the accelerated current increases, the beam "loads" the fundamental mode and must draw its energy from a source appropriately compensated for this "beam loading." Although this problem is especially important for high current superconducting cavities, it is well understood[1] and presents no insuperable problems.

A beam bunch is also capable of exciting other longitudinal and transverse modes in the cavity, and these modes can affect the motion of that bunch, and/or the ones which follow. In an accelerating cavity with sufficiently high current the fields can grow with successive bunches, thereby leading to unacceptably large bunch oscillations (regenerative beam breakup). In a linear accelerator consisting of many identical cavities, the growth of field in one cavity can lead to deflections which provide enhanced excitation of subsequent cavities, thereby also leading to unacceptably large bunch oscillations (cumulative beam breakup). And in circular accelerators or storage rings the cumulative effect can be enhanced by the periodic return of each bunch to the exciting cavity, leading to both longitudinal and transverse beam instabilities.

The present paper is not long enough to review all these and other important beam cavity effects. Instead, we shall (1) summarize the results for transverse regenerative beam breakup, (2) review recent work on transverse cumulative beam breakup of bunched beams in linear accelerators, and (3) show how this recent work can be adapted to the study of transverse instabilities in circular accelerators or storage rings.

Regenerative Beam Breakup

If a transverse mode (a mode excited by off axis motion in the x direction) is excited in a cavity, a beam bunch which enters on axis will experience a deflection, and thereby will be displaced into a region where it may further excite this mode, primarily through the interaction

$$\int dV\, \vec{J} \cdot \vec{E} \simeq N_p ev \int dz\, E_z\big(x(z),0,z;\ t=z/v\big)$$

$$\simeq N_p ev \int dz\, x(z) \left.\frac{\partial E_z}{\partial x}\right|_{x=0} e^{i\omega z/v} \tag{1}$$

where $x(z)$ is the beam bunch trajectory within the cavity. One can express $x(z)$ in terms of the cavity deflecting fields E_x, B_y, and through Maxwell's equations, in terms of $\partial E_z/\partial x$. In this way the increase in field due to the passage of the bunch, which is proportional to the beam current and to the field level itself may be obtained. The decay of the field between bunches is clearly proportional to the field level and to 1/Q. It is therefore easy to see (with due attention to relative phases) that there is a value of average beam current above which the field increase is greater than its decrease, thus leading to a runaway field level and transverse displacement. This value of current is

$$eI_s \simeq 15(p\omega/L^2)/(Z_\perp/L) \tag{2a}$$

where $\omega/2\pi$ is the frequency of the transverse mode and p is the momentum of the (relativistic) beam bunch. The parameter

$$Z_\perp = \frac{2\left|\int_0^L dz\, \frac{\partial E_z}{\partial x}\, e^{i\omega z/v}\right|^2 Q}{\omega\varepsilon \int E^2\, dV} \tag{2b}$$

is the transverse shunt impedance of the cavity whose length is L. Note that, apart from a phase factor, $Z_\perp$ is proportional to L, so that I_s is inversely proportional to L^2.

The starting current for regenerative beam breakup was first derived by P. B. Wilson[2] for traveling wave acceleration. The analogous result given in eq. (2a) applies to acceleration by a standing wave cavity.[2]

Cumulative Beam Breakup

Difference Equations

If one assumes a linear trajectory for each beam bunch as it passes through each cavity, it is possible to derive a set of difference equatons for the transverse displacement $\xi(N,M)$ and angle $\Theta(N,M)$ of the Mth bunch as it enters cavity N. Assuming identical cavities which are electrically uncoupled, and a coasting beam, the equations can be written as

$$\xi(N+1,M) = M_{11}\xi(N,M) + M_{12}[\Theta(N,M) + F(N,M)] \tag{3a}$$

$$\Theta(N+1,M) = M_{21}\xi(N,M) + M_{22}[\Theta(N,M) + F(N,M)] \tag{3b}$$

where

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix} \tag{4}$$

is the usual Courant-Snyder parameterization of the focussing system between cavities and where

$$F(N,M) = \hat{R} \sum_{\ell=0}^{M-1} s_{M-\ell}\, \xi(N,\ell) \tag{5}$$

is the transverse impulse given to bunch M due to the passage of all earlier bunches through cavity N. Here

$$\hat{R} = \frac{N_p e^2 Z_\perp}{2\gamma m_0 c Q} \tag{6}$$

is a parameter proportional to the charge in a bunch (assumed to have negligible longitudinal extent), and to the ratio of the transverse shunt impedance of the mode to its quality factor. The cavity oscillation is contained in the factor

*Permanent address: University of Maryland, College Park, MD 20742; work supported in part by DOE Contract #AS05-80ER10666.

$$s_k = e^{-k\omega\tau/2Q} \sin(k\omega\tau) \qquad (7)$$

where $\omega\tau$ is the phase advance of the transverse mode between bunches.

Exact Solution

Gluckstern, Cooper, and Channell[3] have obtained an explicit solution of eqs. (3) in terms of products of Gegenbauer polynomials, and as an equivalent integral representation. Specifically, they showed that

$$\xi(N,M) = \sum_{m=0}^{M} e^{-m\frac{\omega\tau}{2Q}} \Big[\xi(0,M-m)\ G(x,y) + \sin\mu\ \eta(0,M-m)\ H(x,y)\Big] \quad , \qquad (8)$$

where

$$x = \cos\omega\tau,\ y = \cos\mu,\ \eta(n,m) = \beta\theta(n,m) + \alpha\xi(n,m) \quad , \qquad (9)$$

and

$$G(x,y) = \frac{1}{2\pi i} \oint \frac{du}{u^{m+1}} \left[\cosh(N\sigma) + \frac{J(u)\sinh N\sigma}{\sinh\sigma}\right] \qquad (10)$$

$$= \sum_{j=0}^{\infty} (4\Delta)^j \binom{N+j}{2j} C^j_{m-j}(x)\ C^j_{N-j}(y) \quad , \qquad (11)$$

$$H(x,y) = -\frac{1}{2\pi} \oint \frac{du}{u^{m+1}} \sinh N\sigma \qquad (12)$$

$$= \sum_{j=0}^{\infty} (4\Delta)^j\ C^j_{m-j}(x)\ C^{j+1}_{N-j-1}(y) \quad . \qquad (13)$$

Here

$$\Delta = \frac{\beta\hat{R}}{4} \sin\mu \sin\omega\tau \quad ,$$

$$J(u) = \frac{2\Delta u}{1 - 2ux + u^2} \quad , \quad \cosh\sigma = y + J(u) \quad . \qquad (14)$$

Steady State Solution

For constant $\xi(0, M-m) = \xi_0$, $\eta(0, M-m) = \eta_0$ one can perform the sum over m to $M = \infty$ in eq. (8), as well as the contour integrals in eqs. (10) and (12) to obtain the steady state result

$$\xi(N,\infty) = \xi_0 \cosh N\sigma + (J\xi_0 + \eta_0 \sin\mu) \frac{\sinh N\sigma}{\sinh\sigma} \qquad (15)$$

where

$$\cosh\sigma = \cos\mu + J \qquad (16)$$

and where J can be written as

$$J = \frac{\beta\hat{R}\sin\mu}{8} \cdot \frac{\sin\omega\tau}{\sin^2(\frac{\omega\tau}{2}) + \sinh^2(\frac{\omega\tau}{4Q})} \quad . \qquad (17)$$

The result in eq. (15) is exact and indicates that the solution will either oscillate or grow exponentially as N increases, depending on whether $|\cos\mu + J|$ is less than or greater than 1. In making our estimates, we shall assume that $\mu = 0$ and $|J| \ll 1$. The exponent corresponding to exponential growth then can be approximated as

$$e_0 = N\sigma \simeq N(2J)^{1/2} \quad . \qquad (18)$$

The effect of resonance between the mode frequency and the accelerating frequency is contained in eq. (17). Specifically, for large Q, resonant peaks occur at

$$\omega\tau = 2n\pi(1 + \frac{1}{2Q}) \quad , n=1, 2, \ldots \qquad (19)$$

with exponential growth given by

$$(e_0)_{res} = N\left(\frac{L\hat{R}Q}{2n\pi}\right)^{1/2} \quad , \qquad (20)$$

where $M_{12} = L$ is the spacing between cavities in the absence of external focussing. A similar analysis can be carried through for a modulated input displacement.[3]

Transient Behavior

It is also possible to derive the transient behavior for large M and N from the integral representations in eqs. (10) and (12). For a single displaced pulse, and without external focussing, one can use a saddle-point method to show that

$$\frac{\xi(N,M)}{\xi_0} \simeq \frac{\sqrt{E_1}}{M\sqrt{6\pi}}\ e^{\frac{-M\omega\tau}{2Q} + \frac{3\sqrt{3}}{4}E_1} \cos\left(M\omega\tau - \frac{3}{4}E_1 - \frac{\pi}{12}\right) \qquad (21)$$

where

$$E_1 = (L\hat{R}N^2M)^{1/3} \quad . \qquad (22)$$

For fixed N, the maximum oscillation amplitude occurs when

$$M_{max} \simeq N\left(\frac{3}{4}\right)^{3/4}\left(\frac{Q}{\omega\tau}\right)^{3/2}(L\hat{R})^{1/2} - \frac{5}{2}\left(\frac{Q}{\omega\tau}\right) \qquad (23)$$

and reaches a maximum value

$$\frac{|\xi_{max}|}{\xi_0} = \left(\frac{\omega\tau}{3\sqrt{3}\ \pi Q M_{max}}\right)^{1/2} e^{M_{max}\omega\tau/Q} \qquad (24)$$

which gives close agreement with results of the numerical simulation described in ref. 3. The saddle-point calculation also gives good agreement with the approach to equilibrium in the steady state case, where the numerical simulation shows the rapid development of symmetric oscillations about the steady state value of ξ. Space does not permit a discussion of the adiabatic approximation which can be carried through for accelerated or decelerated beams.[3]

Application to Regenerative Beam Breakup

It is possible to construct a simple model for regenerative beam breakup from the equations for cumulative beam breakup. Equation (3) can be put in the form

$$\xi(N+1,M) - 2\xi(N,M) + \xi(N-1,M) = L\hat{R} \sum_{\ell=0}^{M-1} s_{M-\ell}\ \xi(N,\ell) \qquad (25)$$

where we assume no external focussing. But eq. (25) corresponds to the absence of electrical coupling between adjacent cavities. In the case of regenerative coupling we take all N_0 cells (instead of cavities) to be tightly coupled in our simple mode and replace $\xi(N,\ell)$ on the right side by its average over N from 1 to N_0.* In this way the right side of eq. (25) becomes independent of N and permits as a solution (for large N)

$$\xi(N,M) \simeq a(M)\, N^2 \qquad \overline{\xi(N,M)}^N = a(M)\,\frac{N_0^2}{3} \tag{26}$$

leading to the self consistent equation describing the onset of instability

$$6a(M) = L\,\hat{R}\,N_0^2 \sum_{\ell=0}^{M-1} s_{M-\ell}\, a(\ell) \quad . \tag{27}$$

Neglecting phase factors in $a(\ell)$, $s_{M-\ell}$, one obtains for large Q

$$QL\hat{R}\,N_0^2 = 6\,\omega\tau \tag{28}$$

corresponding to the starting current

$$eI_s = 12(p\omega/\mathcal{L}^2)/(Z_\perp/L) \quad . \tag{29}$$

Note that $Z_\perp/L$ can be replaced by $\hat{Z}_\perp/\mathcal{L}$, where

$$\hat{Z}_\perp = \frac{2\left|\int_0^{\mathcal{L}} dz\,\frac{\partial E_z}{\partial x}\,e^{i\omega z/v}\right|^2 Q}{\omega\varepsilon \int E^2\, dV} \tag{30}$$

corresponds to the definition of $Z_\perp$ for a cavity of length $\mathcal{L}$. Thus the result in eq. (29) agrees quite closely with the more accurate result in eq. (2a).

Our "derivation" clearly ignores all phase factors, including the requirement[2] of the phase slip between the oscillations of the beam bunches and the transverse cavity mode.† But we obtain a surprisingly accurate result in our simple model.

Transverse Instability in Circular Accelerators

The displacement of a beam bunch on its Nth revolution in a circular accelerator, interacting with a cavity is governed by the analogue to eq. (25)

$$\xi(N+1) - 2\cos\mu\,\xi(N) + \xi(N-1) = M_{12}\hat{R}\sum_{\ell=1}^{N} s_\ell\,\xi(N-\ell) \tag{31}$$

It can be seen that

$$\xi(N) = Az^{-N} \tag{32}$$

is a solution for large N provided

*We are clearly neglecting phase differences in the excitation of adjacent cells.

†A more careful treatment, including phase factors, changes the factor 12 in eq. (29) to $\pi^3/2$ for a phase slip of π between the beam and the cavity mode over the length of the cavity, in complete agreement with eq. (2a).

$$z + \frac{1}{z} - 2\cos\mu = M_{12}\hat{R}\sum_{\ell=1}^{\infty} s_\ell\, z^\ell = \frac{M_{12}\hat{R}\sin\omega\tau}{az + \frac{1}{az} - 2\cos\omega\tau} \tag{33}$$

where

$$a = e^{-\omega\tau/2Q} \quad . \tag{34}$$

Equation (33) is a quartic equation for z; the solution for $\xi(N)$ in eq. (32) will be stable provided $|z| \le 1$ for all four roots.

We can obtain an approximate solution for

$$M_{12}\hat{R} \ll 1 \quad ,$$

and find for the roots near $e^{\pm i\mu}$

$$|z_1| = |z_2| \simeq 1 + \frac{\beta\hat{R}\sin\omega\tau\sin\mu}{4\Delta^2}\sinh\frac{\omega\tau}{2Q} \tag{35}$$

and for the roots near $a^{-1}e^{\pm i\omega\tau}$

$$|z_3| = |z_4| \simeq 1 + \sinh\frac{\omega\tau}{2Q}\left[\frac{2}{1 + e^{-\omega\tau/2Q}} - \frac{\beta\hat{R}\sin\omega\tau\sin\mu}{4\Delta^2}\right] \tag{36}$$

where

$$\Delta^2 = \cos^2\omega\tau + \cos^2\mu - 2\cos\mu\cos\omega\tau\cosh\frac{\omega\tau}{2Q} + \sinh^2\frac{\omega\tau}{2Q} \tag{37}$$

We therefore will have stability provided

$$0 \le \frac{\beta\hat{R}}{4\Delta^2}\sin\omega\tau\sin\mu \le \frac{2}{1 + e^{-\omega\tau/2Q}} \quad . \tag{38}$$

This method has been extended to the case of several circulating bunches, several identical accelerating cavities, and several transverse modes in each cavity, including the harmonics of the synchrotron oscillations.[4] Results are obtained for frequency shifts and growth rates which are in agreement with those given by Suzuki and Yokoya[5] for the hollow bunch model using the Vlasov equation. In all likelihood, our method could be adapted to obtain the corresponding results for longitudinal instabilities given by Suzuki and Yokoya for the waterbag model.

References

1. See for example P. Lapostolle and A. Septier, Linear Accelerators, John Wiley and Sons, 1970, cf. P. Wilson, p. 1107.

2. See for example P. B. Wilson, Proc. Summer School, Fermilab (1981) p. 450; R. H. Helm and G. A. Loew, Ref. 1, p. 173; see also R. L. Gluckstern, Brookhaven Report AADD-38 (1964).

3. R. L. Gluckstern, R. K. Cooper, and P. J. Channell, Particle Accelerators (to be published).

4. R. L. Gluckstern and T. Suzuki, KEK Report 83-29, March 1984, Tsukuba, Japan.

5. T. Suzuki and K. Yokoya, Nucl. Instrum. and Meth. 203, 45 (1982).