# SPACE CHARGE LIMITS IN THE ACCELERATION OF INTENSE HOLLOW BEAMS

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## Summary

A model for space charge effects in hollow beams is presented. A particular case is treated for hollow beams accelerated in a rf coaxial structure in which strong transverse focusing is present. The beam emittance is treated in cylindrical coordinates so that the forces in the beam can be calculated with respect to the curved, equilibrium trajectory surface. Analysing the focusing forces about this surface, whose curvature is a characteristic of the strong focusing, enables the forces to be treated in linear approximation. The space charge forces are also treated in a linear approximation by considering the beam bunch as a toroidal ring of elliptical cross-section. The space charge limited currents derived can be used to evaluate conditions under which the rf coaxial structure is a suitable device for producing intense beams.

## Introduction

The rf coaxial structure[1] is a spatially continuous focusing device for the acceleration of high currents in the form of hollow beams. The principle of operation of such a device is in many respects analogous to that of the Radio Frequency Quadrupole accelerator, except that the large annular aperture of the coaxial structure suggests an advantage in the acceleration of higher currents. This paper examines the nature of the space charge forces in a bunched hollow beam that place upper limits on the amount of current that can be transported. The space charge limits in dc hollow beams is also being experimentally investigated at the GSI Darmstadt[2,3]. The procedure followed here is that described by Gluckstern[4] for calculating space charge limits in linear accelerators in which the limiting current occurs when the defocusing due to space charge is just balanced by the restoring, focusing forces in the channel. This procedure has also been applied by Wangler[5], for example, to find current limits in RFQ's.

The analysis here for hollow beams differs in that the space charge is distributed inside a toroid instead of in an ellipsoid as for axial beams. Furthermore, both the focusing forces and the space charge forces act about a reference surface that is displaced from the axis of the system and no longer follows a straight line, as is seen in fig. 1.

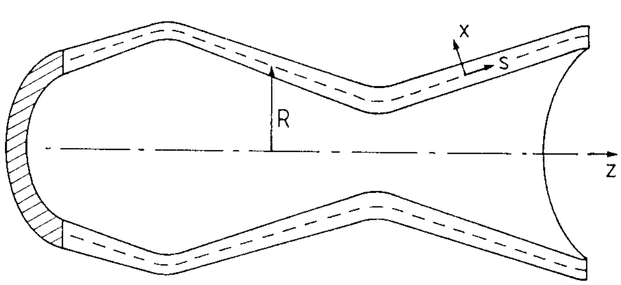


Fig. 1 The beam envelope in a strong focusing channel for hollow beams is modulated. All the forces are calculated with respect to a reference trajectory surface, shown as a dashed line.

As a first step toward calculating the tune depression in a focusing channel resulting from the beam's space charge[6], the emittance of a hollow beam is defined.

## Emittance of a Hollow Beam

It is more reasonable to chose polar coordinates rather than cartesian coordinates to describe the emittance of a hollow beam. If cartesian coordinates are chosen then a hollow beam occupies all x- and y-values over the diameter of the hollow beam and there is no indication of the correlation between x- and y-values that define the annular cross-section of the beam. In polar coordinates, as shown in fig. 2, the beam is described by a radial emittance in the r-r' plane and an azimuthal emittance in the θ-θ' plane. The hollow space in the centre of the beam is evident by the displacement of the radial emittance from the origin in the r-r' plane.

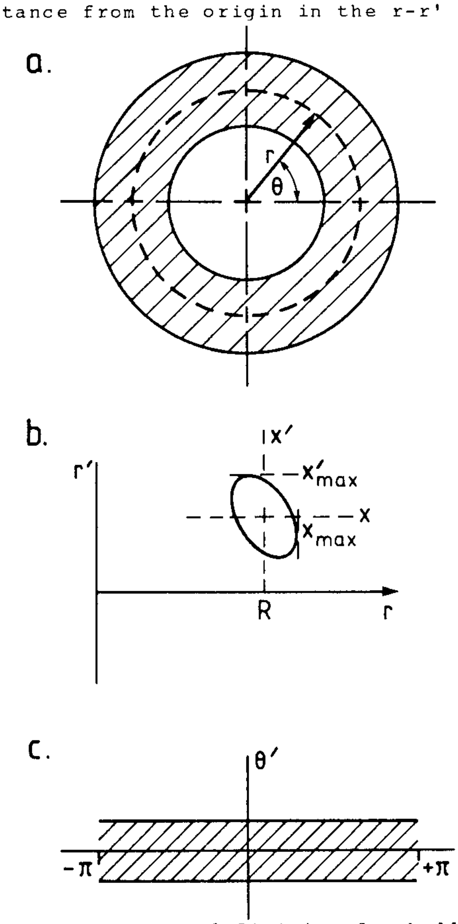


Fig. 2: Emittance definition for hollow beams

a) Polar coordinates in the annular cross-section of the beam
b) Radial emittance measured across a radial section of the beam is represented as an elliptical surface displaced from the origin.
c) Azimuthal emittance measured around the circumference of the beam incorporates the angular momentum of the particles.

To arrive at a correct definition of the four-dimensional transverse emittance one can start with a ribbon beam and describe its emittance in cartesian coordinates, namely

$$\varepsilon_x = \pi \, x_{max} \, x'_{max}$$
$$\varepsilon_y = 4 \, y_{max} \, y'_{max} \qquad (1)$$

If the ribbon beam is now curved back on itself to form a beam of annular cross-section the emittance $\varepsilon_x$ measured across the width of the beam remains the same but the emittance $\varepsilon_y$ must be measured around the circumference of the annulus. Changing to polar coordinates we have $y_{max} = R\,\theta_{max}$ and $y'_{max} = R\,\theta'_{max}$ where for a full annular beam $\theta_{max} = \pi$. In polar coordinates the transverse emittance of a hollow beam is thus

$$\varepsilon_r = \pi \, x_{max} \, x'_{max}$$
$$\varepsilon_\theta = 4 \, \pi \, R^2 \, \theta'_{max} \qquad (2)$$

The quantity $\varepsilon_r$ is the area of the ellipse in the r-r' plane shown in fig. 2b and is a constant of the motion in systems in which there is no coupling between the radial and azimuthal motions (i.e. in the absence of longitudinal magnetic fields.). The azimuthal emittance $\varepsilon_\theta$ is also a constant of the motion and is consistent with the conservation of angular momentum of the individual particles.

## Space Charge Forces in a Hollow Beam

The effect of space charge on a hollow beam differs according to whether an electrode is placed on the axis of the system or not. In the absence of any electrodes the space charge forces will only act radially outwards and cause the beam to grow in diameter. However, an electrode at ground potential, for example, at the centre of the beam can alter the potential variation across the beam so that the space charge forces no longer act to change the average diameter of the beam but cause the width of the annular cross-section of the beam to grow. The latter case is of interest here because we are concerned with hollow beams in strong focusing systems where the inner space of the beam is always occupied by an electrode. In our modelling of the space charge forces we therefore consider them to act linearly about the midsurface of the beam.

In a hollow beam that is also longitudinally bunched we assume that the charge is uniformly distributed in a toroid whose radial cross-section is an ellipse, as shown in fig.3. The space charge density in such a toroid is

$$\rho = \frac{\lambda I}{2 \, \pi^2 \, c \, Rab} \qquad (3)$$

when the current in the linear accelerator is I, the rf wavelength is $\lambda$ and the dimensions R, a and b are as defined in fig. 3.

The electric fields, in the frame of reference of the bunch, due to the space charge can be written

$$E_x = \frac{\rho \, x \, M_x}{\varepsilon_o \, \gamma^2} \; ; \quad E_z = \frac{\rho \, z \, M_z}{\varepsilon_o} \qquad (4)$$

and $E_r$, the electric field which causes the diameter of the toroid to grow, is zero due to the presence of axial conductors. Here $M_x$ and $M_z$ are geometrical form factors for the toroid and are a function of the ratio b/a of the semiaxes of the elliptical cross-section. They satisfy the relation

$$M_x + M_z = 1 \qquad (5)$$

and can be approximated, for a < 2b, by

$$M_z \approx \frac{a}{2b} \qquad (6)$$

The transverse eq. of motion of the particles under the influence of space charge forces only is

$$\gamma m_o \, c^2 \, \beta^2 \, \frac{d^2 x}{d s^2} = q \, E_x \qquad (7)$$

Since the space charge acts to defocus the beam we can use eq. (7) to define a focusing phase advance per $\beta\lambda$ cell, which, with the help of eq. (3), (4) and (5), is given by

$$\sigma_{sc_t}^{\;2} = \frac{- \, q \, I \lambda^3 \, (1-M_z)}{2 \, \varepsilon_o \, c \, \pi^2 \, m_o \, c^2 \, \gamma^3 \, \bar{X} \, b \, \bar{R}} \qquad (8)$$

Defining the space charge defocusing in this way allows us to compare its magnitude with the restoring, focusing forces in an accelerator. The transverse space charge factor is defined as the ratio of these two quantities

$$\mu_t = \frac{\sigma_{sc_t}^{\;2}}{\sigma_{o_t}^{\;2}} \qquad (9)$$

The transverse focusing that remains after the space charge depression is now

$$\sigma_t^{\;2} = \sigma_{o_t}^{\;2} - \sigma_{sc_t}^{\;2} \qquad (10)$$

so that $\mu_t$ can also be written as

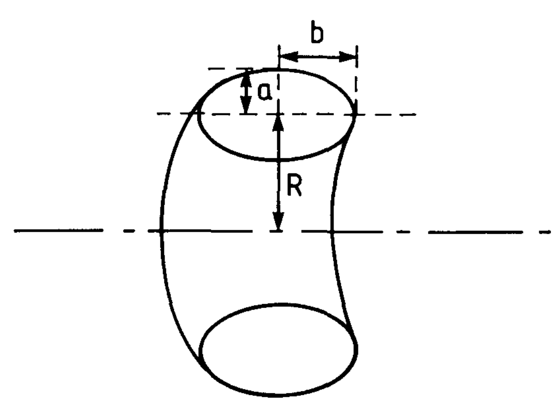


Fig. 3 Space charge forces in a bunched hollow beam are analysed by considering the charge to be uniformly distributed inside a toroid having an elliptical cross-section.

$$\mu_t = 1 - \frac{\sigma_t^{\,2}}{\sigma_{o_t}^{\,2}} \qquad (11)$$

$$= 1 - \frac{\varepsilon_r^{\,2}}{\alpha_r^{\,2}}$$

Similarly, for the longitudinal defocusing due to space charge,

$$\sigma_{sc_l}^{\,2} = \frac{-\,q\, I\, \lambda^3\, M_z}{2\, \varepsilon_o\, c\, \pi^2\, m_o c^2\, \gamma^3\, \bar{X}\, b\, \bar{R}} \qquad (12)$$

and the longitudinal space factor is

$$\mu_l = \frac{\sigma_{sc_l}^{\,2}}{\sigma_{oz}^{\,2}} \qquad (13)$$

## Focusing Forces Acting on a Hollow Beam

In a coaxial structure[1] the radial electric field provides strong focusing for hollow beams. The zero-current transverse focusing phase advance per cell is given by

$$\sigma_{oo} \approx \frac{q\, V\, \lambda^2\, \chi}{4\, \pi\sqrt{2}\, m_o c^2\, \gamma\, R\, a} = \frac{q\, E_{max}\, \lambda^2\, \chi}{2\pi\, \sqrt{2}\, m_o c^2\, \gamma\, R} \qquad (14)$$

where the width of the minimum clear aperture between the inner and outer electrodes is 2a and $E_{max}$ is the maximum electric field at the surface of the inner conductor. The quantity $\chi$ is a focusing efficiency factor whose value $\leq 1$ and depends on the amplitude of the modulations on the electrodes which produce the accelerating field.

Longitudinal restoring forces on the bunch are supplied by the accelerating field. The modulations on the coaxial structure provide a longitudinal electric field of magnitude $E_oT$ along the midsurface of the hollow beam where $T = \pi/4$ is the transit time factor and $E_o$ is the amplitude of the longitudinal field. Since this quantity is independent of the diameter of the hollow beam, the longitudinal motion of the particles is described in exactly the same way as in conventional linear accelerators. The longitudinal focusing phase advance is

$$\sigma_{oz}^{\,2} = \frac{-2\pi\, q\, E_oT\, \sin\phi_s\, \lambda}{m_o c^2\, \beta\, \gamma^3} \qquad (15)$$

where $\phi_s$ is the synchronous phase angle for the structure. Phase stability requires that $\phi_s$ be negative which implies that transverse defocusing will occur as a consequence of the rf acceleration. The transverse focusing strength that includes this rf defocusing is

$$\sigma_{o_t}^{\,2} = \sigma_{oo}^{\,2} - \sigma_{oz}^{\,2} \qquad (16)$$

This transverse focusing confines the transmitted beam within the minimum clear aperture of the system. In order to relate the average beam envelope size $\bar{X}$ in eq. (8) to the aperture half-width a we make use of the envelope modulation factor defined as

$$\psi = \left(\frac{\beta_{max}}{\beta_{min}}\right)^{1/2} = \frac{X_{max}}{X_{min}} = \frac{R_{max}}{R_{min}}$$

The envelope modulation factor is independent of space charge, as is seen, for example, in the form of the betatron function given by Wangler[5] where $\psi$ can be calculated as a function of $\sigma_{o_t}$.

As a result of the strong focusing, the midsurface of the hollow beam is deflected between $R_{max}$ and $R_{min}$ so that the effective aperture available to the beam is

$$a' = a - \bar{R}\, \frac{\psi - 1}{\psi + 1} \qquad (17)$$

If the structure is carrying the maximum current determined by the limit of the transverse focusing, then the maximum in the beam envelope $X_{max}$ will just fill the available aperture a'. The mean value for the envelope used in eq. (8) can then be written as

$$\bar{X} = (X_{max}\, X_{min})^{1/2} = \frac{a'}{\psi^{1/2}} \qquad (18)$$

The maximum bunch length, 2b, is found from the limits of the phase stable area. The phase stable area is reduced under the influence of space charge so that

$$b = \frac{3\, \beta\, \lambda\, |\phi_s|\, (1-\mu_l)}{4\pi} \qquad (19)$$

If $\mu_l$ is chosen to be equal to 1/3 it will be seen that the longitudinal current is at a maximum, and for this special case of a fixed longitudinal space charge parameter we get

$$b = \frac{\beta\lambda\, |\phi_s|}{2\pi} \qquad (20)$$

## Current Limits

If the limits to the bunch dimensions defined in eq. (18) and (19) are used in eq. (8) the current in this equation becomes the transverse space charge limited current. Using (9) this current limit can be written

$$I_t = \frac{3\varepsilon_o c\, \pi\, m_o c^2\, \gamma^3\, a'\bar{R}\, \beta\, |\phi_s|\, (1-\mu_l)\, \mu_t\, \sigma_{o_t}^{\,2}}{2\, q\, \lambda^2\, \psi^{1/2}\, (1-M_z)} \qquad (21)$$

The expression is applicable in situations where restrictions are placed on $\sigma_{o_t}$, as is the case, when $\sigma_{o_t} < \pi/2$ to avoid resonances and instabilities[7]. However, it is more likely that the restrictions on $\sigma_{o_t}$ will occur first as a result of maximum electric field strengths imposed by sparking limits. Incorporating eq. (14) and (16) gives a transverse current limit

$$I_t = \frac{3\, \varepsilon_o\, c\, q\, \gamma\, a'\, \beta\, |\phi_s|\, \lambda^2\, E_{max}^{\,2}\, \chi^2\, (1-\mu_l)\, \mu_t}{16\, \pi\, m_o c^2\, \bar{R}\, \psi^{1/2}\, (1-M_z)} \cdot \left(\frac{\sigma_{oo}^{\,2} - \sigma_{oz}^{\,2}}{\sigma_{oo}^{\,2}}\right) \qquad (22)$$

Following the same procedure for the longitudinal space charge in eq. (12) gives the longitudinal current limit,

$$I_l = \frac{3\, \varepsilon_o\, c\, \pi^2\, \bar{R}\, a'\, E_oT\, |\phi_s|\, \sin\phi_s\, \mu_l\, (1-\mu_l)}{\lambda\, \psi^{1/2}\, M_z} \qquad (23)$$

Using the approximation for $M_z$ in eq. (6) this becomes

$$I_l \approx 4.5\, \frac{\varepsilon_o\, c\, \pi\, R'\, \beta\, E_oT\, |\phi_s|^2\, \sin\phi_s\, \mu_l\, (1-\mu_l)^2}{} \qquad (24)$$

which has a maximum value when $\mu_1 = 1/3$, leading to the commonly used expression for the bunch length in eq. (20).

## Discussion

The essential difference to conventional, axial beam accelerators that emerges from this analysis is the dependence of the beam current limits on the size of the hollow beam radius $\bar{R}$. In the case where an upper limit is placed on $\sigma_{o_t}$, the current increases linearly with the aperture a' and radius $\bar{R}$ as in eq. (21). However, when the upper limit is determined by the maximum allowable field strength, then the current is proportional to $a'/\bar{R}$. The longitudinal current limit, on the other hand, always increases in proportion to $\bar{R}$. The optimum hollow beam radius at which the transverse and longitudinal current limits are equal is found by equating (22) and (23), giving

$$\bar{R}^2 = \frac{q\,\gamma\,\beta\,\lambda^3\,E_{max}^2\,\chi^2}{16\pi^3\,m_oc^2\,E_oT\,\sin\phi_s}\,\left(\frac{\sigma_{oo}^2-\sigma_{oz}^2}{\sigma_{oo}^2}\right)\cdot\frac{M_z}{1-M_z}\,\frac{\mu_t}{\mu_1} \qquad (25)$$

Using this value for $\bar{R}$ in either eq. (22) or (23) gives an expression for the maximum current when the longitudinal and transverse limits are equal.

$$I_{max} = \frac{3\,\varepsilon_o\,c\,(\pi\,q\,\gamma\,\beta\,\lambda\,E_oT\,\sin\phi_s)^{1/2}\,a'\,|\phi_s|\,E_{max}}{4\,(m_oc^2\,\psi\,M_z\,(1-M_z))^{1/2}}\cdot\left(\frac{\sigma_{oo}^2-\sigma_{oz}^2}{\sigma_{oo}^2}\right)^{1/2}(\mu_t\,\mu_1)^{1/2}\,(1-\mu_1) \qquad (26)$$

which also has a maximum for $\mu_1 = 1/3$. The transverse space charge factor should, on the other hand, be chosen as close to 1 as possible before running into undesirable non-linear effects. Most of the remaining parameters in eq. (26) are determined by practical considerations arising from the area of application of the accelerator. The current limit is also strongly dependant on the maximum field strengths that can be tolerated in the structure. An upper limit will be placed on the value of the transverse space charge factor $\mu_t$ according to the quality of the beam emittance and the degree of tune depression that is tolerable. The main task of the designer, then, is to optimise the focusing strengths so that the term

$$\frac{a'}{\psi^{1/2}}\left(\frac{\sigma_{oo}^2-\sigma_{oz}^2}{\sigma_{oo}^2}\right)$$

in eq. (26) is a maximum.

Evaluating eq. (25) and (26) gives a guide as to the suitability of using hollow beam geometry in a given application. Other values for the beam radius may be chosen than that dictated by eq. (25), which will lower the current from its optimized maximum. Practical accelerator designs, however, usually involve some compromises. This analysis gives a useful indication of the dependance of the beam current on the parameters of the beam and the structure, but it should not be forgotten that it is an approximation which does not include the many non-linear effects that can arise in an accelerator.

## Acknowledgements

Thanks are extended to R.W. Müller for many discussions on the topic of space charge limits in accelerators; also to H.G. Hereward for his critical reading of the manuscript and suggested changes.

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