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LUMPED-CIRCUIT MODEL OF FOUR-VANE RFQ RESONATOR*

Thomas P. Wangler, AT-1, MS-H817
Los Alamos National Laboratory, Los Alamos, NM 87545 USA

Summary

Although the rf cavity code SUPERFISH[1] is a necessary tool for designing rf cavities, it is often useful to have approximate analytic formulas for the electromagnetic properties of a cavity. One approach for the RFQ four-vane cavity is to use the analytic solutions associated with an inclined plane waveguide.[2]

In this paper, we make use of the result that the large capacitive vane loading in the four-vane RFQ resonator allows a convenient representation by a simple lumped-circuit model. Formulas are derived that depend on a single unknown parameter: the vane capacitance per unit length, which can be calculated for different vane geometries using SUPERFISH. The formulas from the model are useful for estimating the RFQ's electromagnetic properties as a function of parameters such as frequency and intervane voltage.

Lumped-Circuit Model

Figure 1 shows the cross section of an idealized four-vane resonator consisting of four identical quadrants. In the TE_{210}-like quadrupole mode, the vane-tips are charged by currents that flow in the quadrant walls to produce an intervane voltage V as shown in the figure. Also shown are the electric field lines for this mode, localized near the vane tips. The magnetic field, which is parallel to the central beam axis (perpendicular to the plane of fig. 1), is predominantly in the outer quadrants and is 90° out of phase with respect to the electric field. The whole field pattern varies sinusoidally in time and, for unmodulated vanes, is independent of longitudinal position. In the following discussion, we will ignore effects associated with vane modulation.

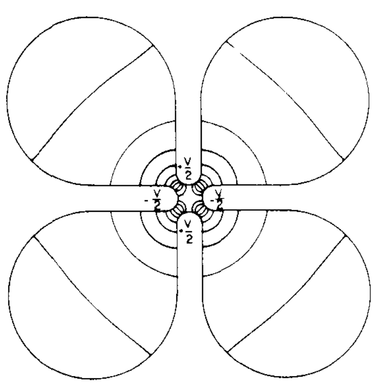

Fig. 1. Cross section of the RFQ four-vane resonant cavity. The electric lines of force and vane-tip voltages are shown.

The quadrupole mode can be characterized by specifying that the electric field be parallel to the horizontal and vertical lines that pass through the central beam axis and through the vane tips of like polarity. Then each of the four quadrants may be analyzed as an independent resonant cavity, subject to this boundary condition. The geometry of a single quadrant is shown in fig. 2.

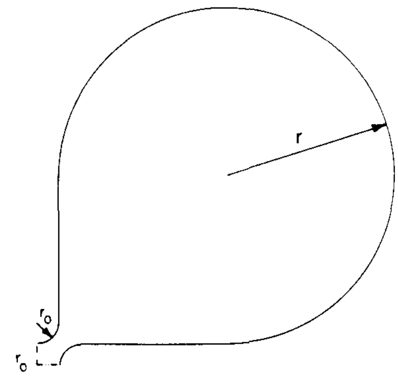

Fig. 2. Geometry of one RFQ quadrant.

In the lumped-circuit model, we will assume that each quadrant of length ℓ can be represented by a lumped capacitance C' and a lumped inductance L', each of which is associated with the localized regions of electric and magnetic field, respectively. For the total cavity, this results in the equivalent circuit for the quadrupole mode shown in fig. 3. A more general form of this equivalent circuit that included both dipole and quadrupole modes was suggested earlier by Potter.[3] The four resonant quadrants provide separate electrical paths in parallel between points of maximum positive and negative potential. Therefore, if C_ℓ is the total capacitance per unit length and ℓ is the cavity length, we may write $C_\ell = 4C'/\ell$.

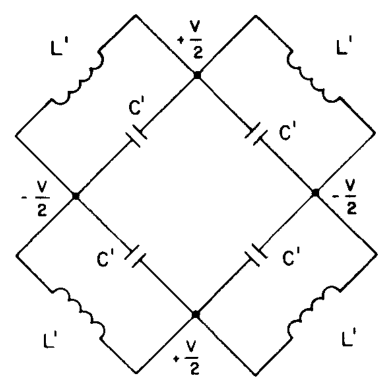

Fig. 3. An equivalent circuit to represent the transverse RFQ electromagnetic properties.

For a single quadrant as shown in fig. 2, we will compute the inductance as follows. We use $\oint \vec{B}\cdot d\vec{\ell} = \mu_0 I$ in the outer part of the cavity where the electric field is small and take a path parallel to the beam axis near the conducting surface that returns inside the conductor where B is zero. Then $B = \mu_0 I/\ell$, where I is the transverse current over a length ℓ. If the magnetic field is approximately uniform in the outer region of the cavity, we write the magnetic flux in each quadrant as $\phi = BA = \mu_0 AI/\ell$, where A is the effective cross-sectional area of a quadrant. The inductance for each quadrant of length ℓ is then a ratio of flux to current, or $L' = \mu_0 A/\ell$. To develop the model further, we assume the effective quadrant area consists of three-quarters of a circle of radius r plus a square

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with sides of length r as shown in fig. 4. Then, the area is $A = (4 + 3\pi)r^2/4$ and the quadrant inductance is $L' = (4 + 3\pi)\mu_0 r^2/4\ell$. The cavity resonant frequency

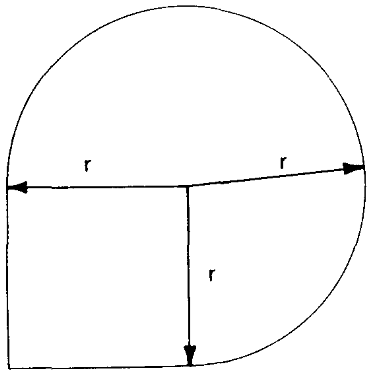

Fig. 4. Shape of idealized quadrant cross-sectional area assumed for inductance calculation.

is given by $\omega^2 = (L'C')^{-1}$. If we substitute for L' and C', we obtain a relation between quadrant radius, frequency, and capacitance per unit length, which is

$$r^2 = \left(\frac{16}{4 + 3\pi}\right) \frac{1}{\mu_0 C_\ell \omega^2} \quad . \tag{1}$$

Thus, increased capacitive loading or higher frequency reduces the quadrant radius and, likewise, the transverse dimensions of the overall cavity.

For an $e^{j\omega t}$ time dependence of the currents and voltages, the peak current on the outer wall is given by $I = j\omega C_\ell \ell V/4$, where V is the peak intervane voltage. Then, the peak values of the current per unit length I_ℓ on the outer wall and magnetic field are

$$I_\ell = j\omega C_\ell V/4 \tag{2}$$

and

$$B = j\mu_0 \omega C_\ell V/4 \quad . \tag{3}$$

We calculate the power loss by assuming the conducting surface area for each quadrant consists of the cross section of fig. 4 taken over a length ℓ, giving a total surface area of $S = (4 + 3\pi)2r\ell$, which includes all four quadrants. The power loss is

$$P = (1/2)R_s(B/\mu_0)^2 S,$$

where R_s is the surface resistance. If we use eq. (3) for B and use eq. (1) to eliminate r, we obtain a power per unit length for the whole RFQ cavity of

$$P_\ell = \left[\frac{4 + 3\pi}{32\sigma}\right]^{1/2} (\omega C_\ell)^{3/2} V^2 \quad , \tag{4}$$

where we have used $R_s = (\mu_0 \omega/2\sigma)^{1/2}$ and σ is the conductivity. The stored energy per quadrant is $W' = 1/2(C'V^2)$, which implies a stored energy per unit length for the whole RFQ cavity of

$$W_\ell = \frac{1}{2} C_\ell V^2 \quad . \tag{5}$$

The quality factor is $Q = \omega W_\ell/P_\ell$, which can be rewritten using eqs. (4) and (5) as

$$Q = \left(\frac{8\sigma}{(4 + 3\pi)\omega C_\ell}\right)^{1/2} \quad . \tag{6}$$

A local effective shunt impedance per unit length can be calculated from a definition $ZT^2 = (E_0T)^2/P_\ell$, using eq. (4) and using $E_0T = \pi AV/2\beta\lambda$, where A is the acceleration efficiency,[4] β is the synchronous particle velocity, and λ is the free space wavelength. The result is

$$ZT^2 = \left(\frac{\sigma}{8(4 + 3\pi)}\right)^{1/2} \left(\frac{A}{\beta c}\right)^2 \frac{\omega^{1/2}}{C_\ell^{3/2}} \quad , \tag{7}$$

where c is the speed of light.

The acceleration efficiency A varies typically from 0 to about 0.5. The effective shunt impedance as given by eq. (7) varies throughout the RFQ as a result of the variations in A and β. The usefulness of this quantity is probably restricted to the accelerator section of an RFQ, where acceleration rate is an important performance criterion. It may be a useful parameter for deciding on the correct transition energy for injection into a structure that follows the RFQ. The above formulas are expressed in SI units. The frequency in our formulas should be expressed as $\omega = 2\pi f$ in hertz and voltage V is in volts. This will give I_ℓ in amperes/meter, B in tesla, W_ℓ in joules/m, P_ℓ in watts/meter, and ZT^2 in ohms/meter. For room-temperature copper, we use $\rho = \sigma^{-1} = 1.7 \times 10^{-8}\ \Omega \cdot m$ and $\mu_0 = 4\pi \times 10^{-7}\ kg \cdot m/C^2$.

The formulas are repeated below with numerical constants for a room-temperature copper cavity. Substitute V in volts, C_ℓ in farads/meter, and f in hertz.

$$I_\ell = j\ 1.6\ fC_\ell V \quad (A/m), \tag{8}$$

$$B = j\ 2.0 \times 10^{-6}\ fC_\ell V \quad (tesla), \tag{9}$$

$$P_\ell = 1.3 \times 10^{-3}(fC_\ell)^{3/2}V^2 \quad (W/m), \tag{10}$$

$$W_\ell = 0.5\ C_\ell V^2 \quad (J/m), \tag{11}$$

$$Q = 2.4 \times 10^3 (fC_\ell)^{-1/2} \quad , \text{ and}$$

$$ZT^2 = 2.1 \times 10^{-14}(A/\beta)^2 f^{1/2} C_\ell^{-3/2} \quad (\Omega/m) \quad . \tag{12}$$

The power per unit-length formula includes only the RFQ four-vane cavity. In some cases, beam power or an RFQ manifold may also contribute to the total power. Furthermore, P_ℓ, Q, and ZT^2 do not include end-plate losses that, however, are usually a small fraction of the total.

Capacitance Per Unit-Length Values

The formulas presented in the previous section depend upon one unknown parameter: the effective capacitance per unit length C_ℓ. Electrostatic calculations show that, as would be expected, C_ℓ is independent of radial aperture for four poles of circular cross section whose radius of curvature is equal to the radial aperture. The result obtained for circular poles is about $C_\ell = 90 \times 10^{-12}$ farads/meter.[5] A larger value, together with an associated weak dependence on radial aperture, might be expected for vanes rather than circular poles.

The computer program SUPERFISH was used to calculate the electromagnetic properties of the cavity with the geometry shown in fig. 2. The resonant frequency was calculated as a function of the radius r and radial aperture r_0. The vane-tip radius of curvature was kept equal to the radial aperture r_0 for these studies. Using the radius r and the resonant frequency calculated from SUPERFISH, C_ℓ was calculated for each case using eq. (1). Figure 5 shows that the calculated C_ℓ is a rather weak function of the dimensionless parameter r_0/λ, as expected. A decrease of C_ℓ as r_0/λ increases is expected as the contribution of vane sides decreases relative to a fixed contribution from the vane tips. For $0.002 \leq r_0/\lambda < 0.008$, we obtain C_ℓ values in farads/meter that can be approximated by the formula $C_\ell = 48 \times 10^{-12}(r_0/\lambda)^{-1/6}$. An alternate prescription to determine C_ℓ is to use eq. (5) with W_ℓ and V from the SUPERFISH runs. This prescription yields C_ℓ values about 3% lower than the curve in fig. 5. Thus, C_ℓ depends somewhat on the prescription chosen; therefore, the numerical values given in fig. 5 must not be considered as absolute.

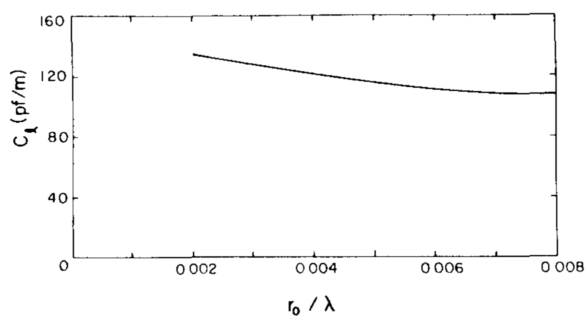

Fig. 5. Capacitance per unit length versus r_0/λ deduced using SUPERFISH calculations with a procedure given in the text. C_ℓ may be described by the approximate formula $C_\ell = 48 \times 10^{-12}$ $(r_0/\lambda)^{-1/6}$ in farads/meter.

The values of C_ℓ given in fig. 5 have been used to evaluate B, P_ℓ, W_ℓ, and Q from eqs. (3)-(6) for comparison with the more correct SUPERFISH results. For the cases studied, we have found that the P_ℓ values agree to better than 10%, whereas the B, W_ℓ, and Q values agree to better than 5%. For a given intervane voltage and with C_ℓ given from fig. 5, the model overestimates B, W_ℓ, and P_ℓ and underestimates Q.

The model can be useful for estimating cavity properties in the early stages of an RFQ design as well as for showing the dependence of the cavity properties on ω, C_ℓ, and V. A more detailed SUPERFISH study could be made to evaluate C_ℓ for more realistic vane geometries.

Field, Stored Energy, and Power Scaling with Frequency

Except for Q and ZT^2, which are independent of cavity excitation, the formulas in the previous section have been expressed in terms of intervane-voltage V. Often the RFQ voltage is limited by the peak surface electric field E_s. Because the transverse resonator dimensions are proportional to rf wavelength, and $E_s \propto V/r_0$ (a result verified by SUPERFISH calculations), then $V \propto E_s/\omega$. This implies that

$$I_\ell \propto C_\ell E_s \quad ,$$

$$B \propto C_\ell E_s \quad ,$$

$$P_\ell \propto C_\ell^{3/2} E_s^{\,2}/\omega^{1/2} \quad , \text{ and}$$

$$W_\ell \propto C_\ell E_s^{\,2}/\omega^2 \quad .$$

If E_s is independent of frequency, we would conclude that I_ℓ and B are independent of frequency and P_ℓ and W_ℓ decrease as frequency increases. However, often a Kilpatrick-criterion[6] scaling is used for RFQ design, which gives the peak surface field as $E_s = bE_K$, where

$$f(\text{MHz}) = 1.643\, E_K^{\,2} e^{-8.5/E_K} \quad , \tag{13}$$

with E_K in megavolts/meter and b is a bravery factor to be chosen by the designer (typically b = 1.5 to 2.0). An approximate representation for eq. (13) in the 100- to 1000-MHz frequency range of interest for many RFQ designs is given by

$$E_K(\text{MV/m}) = 1.8\, f(\text{MHz})^{0.4} \quad . \tag{14}$$

Then the following approximate scaling relations are obtained:

$$I_\ell \propto C_\ell \omega^{0.4} \quad ,$$

$$B \propto C_\ell \omega^{0.4} \quad ,$$

$$P_\ell \propto C_\ell^{3/2} \omega^{0.3} \quad , \text{ and}$$

$$W_\ell \propto C_\ell/\omega^{1.2} \quad .$$

We see that I_ℓ and B, which are proportional to E_s, will increase with frequency; P_ℓ also increases rather weakly with frequency, but W_ℓ still decreases with frequency.

Acknowledgments

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