STABILITY AND EMITTANCE GROWTH OF DIFFERENT PARTICLE PHASE SPACE DISTRIBUTIONS IN PERIODIC QUADRUPOLE CHANNELS

J. Struckmeier, J. Klabunde, GSI Darmstadt
M. Reiser, University of Maryland

Abstract

The behavior of K-V, waterbag, parabolic, conical, Semi-Gaussian and Gaussian transverse phase space distributions in periodic quadrupole channels has been studied by particle simulations. Phase space boundaries that are initially ellipsoidal are used except for the Semi-Gaussian distribution, where the boundary is rectangular. It has been found that all of these distributions exhibit the known K-V instabilities. However, the action of the K-V type modes becomes increasingly damped in the order of the distributions quoted above.

In addition, the distributions with non-homogeneous charge density in real space experience, in only one period of the channel, a rapid initial emittance growth. This growth, which becomes very significant at high beam intensities, can be attributed to the homogenization of the charges in real space, resulting in a conversion of electric field energy into transverse kinetic and potential energy. This effect arises because neither the ellipsoidal phase space symmetry is conserved for the non-K-V distributions if the space charge forces are not negligible, nor the rectangular symmetry if the space charge forces are not infinite. Two simple analytical formulae have been derived to estimate the upper and lower boundary values for this effect and are compared with the results obtained from particle simulations.

For a homogeneous point density of the phase space (usually called a "waterbag" distribution), a stationary distribution for continuous focusing can be constructed by choosing the phase space boundary properly. A subsequent canonical transformation leads at least to a quasi-stationary waterbag distribution for periodic (i.e. non-continuous) focusing.

1. Third-Order Instability with Equivalent Beams of Different Particle Distributions

The space charge potential of a K-V distribution is a quadratic function of the spatial coordinates leading to linear forces. As has been shown previously[1], the K-V distribution is unstable against perturbations of this potential in some specific regions depending on the external focusing forces (defined by the phase advance σ_0) and the beam parameters (defined by σ). The type of perturbation can be classified by the order of the additional potential terms. For $\sigma_0=90°$, the K-V distribution is unstable against "third order" perturbations in the region $38° \leq \sigma \leq 60°$. To evaluate whether and how this instability affects the non-K-V distributions that have some specific density profiles in phase space, we performed simulation studies for $\sigma_0 = 90°$ and $\sigma = 41°$. The maximum growth rate for the third-order mode in a K-V beam occurs near this condition. All computer runs discussed in this paper were made with the parameters of the GSI magnetic quadrupole channel (see Ref. 2).

Fig. 1 shows the beam evolution starting with a K-V distribution. The specific instability can easily be recognized by the three "arms" growing out of the initial elliptic particle distribution. After about 50 periods the transverse r.m.s. emittance has grown by a factor of 2.2. It remains nearly constant during the following sections.

Starting with equivalent (i.e. the r.m.s. values are the same) waterbag and parabolic transverse phase space distributions and transforming them under the same conditions, the resulting type of distortion is the same, but the growing of "arms" is less pronounced. The r.m.s. emittance growth factors of 2.2 after 100 periods are essentially the same as that of the K-V distribution.

The evolution of the conical distribution shows a less pronounced third-order instability mode compared to the previous types of distributions. The growth factor of the r.m.s. emittance is lower, reaching a value of ≈1.8 after 100 periods.

Nearly no third-order mode patterns are observed for the evolution of equivalent initial Semi-Gaussian (or "thermal", i.e. a homogeneous particle density in real space and a Gaussian in velocity space) and Gaussian distributions. A nearly constant small increase in r.m.s. emittance is obtained. No saturation of the emittance growth can be recognized after 100 periods. At that point, the growth factors of the r.m.s. emittance are only ≈1.2 .

Fig. 3 shows a plot of the transverse r.m.s. emittance growth factors versus the number of cells for the six distribution functions. The initial "offset" of the distributions having no homogeneous charge density in real space is due to the "homogenization" effect of the particle density in that space (see discussion in section 2).

The "saturation" of the emittance growth occurs at a value of ≈2.3, except for the conical and the two types of Gaussian distributions, which continue to increase but at lower levels. The slope of emittance growth decreases from the K-V towards the Gaussian distributions. Therefore the number of cells after which a saturation can be observed, if there is any, increases. This is due to the increasing spread in the individual particle tunes σ, which minimizes the effect of the resonance. This effect remains to be investigated.

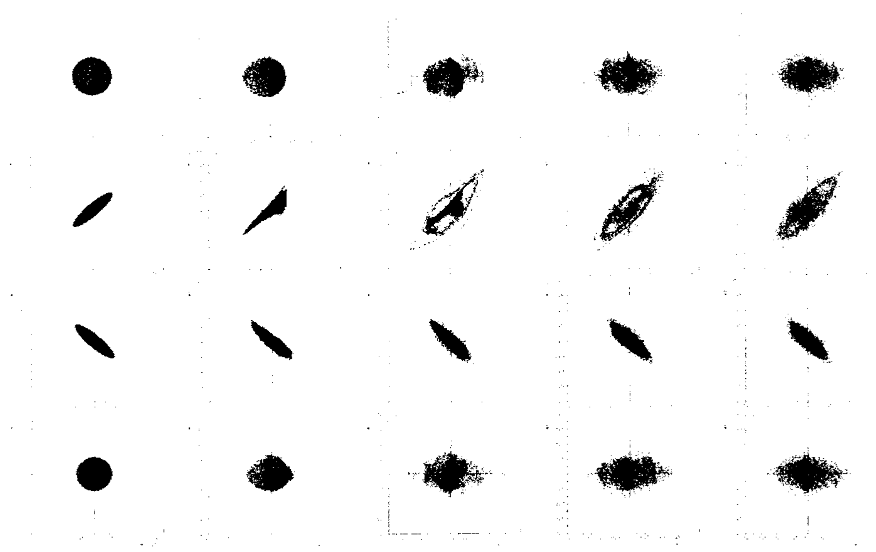

Fig. 1 Transformation of an initial K-V distribution through the GSI FODO channel at $\sigma_0=90°$, $\sigma=41°$

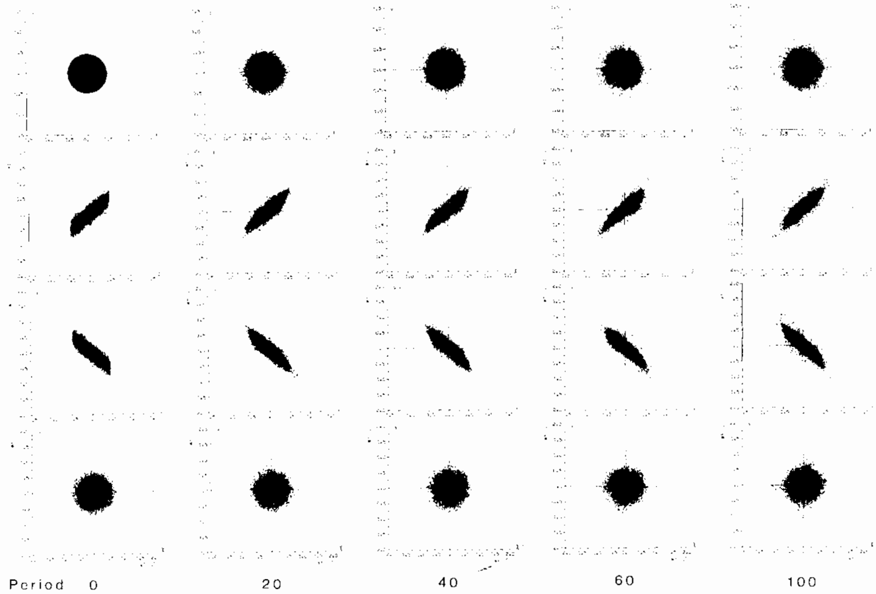

Fig. 2 Transformation of an initial Semi-Gaussian distribution through the GSI FODO channel at $\sigma_0=90°$, $\sigma=41°$

2. Initial Emittance Growth of non-stationary Particle Distributions

The ellipsoidal phase space symmetry of the non-K-V distributions is conserved only for zero current beam transport. Therefore, an internal redistribution of the phase space points occurs if these distributions are transported under space charge conditions. This redistribution leads to a more homogeneous charge density in real space, whereby field energy is converted into transverse kinetic and potential beam energy. It also means that a certain amount of transverse r.m.s. emittance growth occurs. This "homogenization" of the particle density in real space takes place within the first cell as can be seen in Fig. 4 for $\sigma_0=60°$, $\sigma=15°$. No structure resonance is present under these conditions. Fig. 5 shows the increase of the r.m.s. emittance versus the cell number for this case. The size of the "initial" emittance growth depends on the type of initial distribution. It increases in going from the waterbag over the parabolic and conical towards the Gaussian distribution. For each distribution we calculated a dimensionless geometry factor "f", which defines the difference between the field energy of the specific charge distribution in real space and that of a homogeneous one (as it is for the K-V- and the Semi-Gaussian types). Assuming that this difference is transformed into kinetic and potential energy and then into kinetic energy only via homogenization of the charge density in real space, one obtains the following estimations for the "initial" emittance growth[3]:

$$\sqrt{1 + f\cdot\left(\frac{\sigma_0^2}{\sigma^2} - 1\right)} \; < \; \frac{\langle\varepsilon\rangle^*}{\langle\varepsilon\rangle} \; < \; \sqrt{1 + 2f\cdot\left(\frac{\sigma_0^2}{\sigma^2} - 1\right)}$$

These formulae estimate an initial emittance growth of 4.1% to 8.1% for the waterbag, 8.5% - 16.3% for the parabolic, 10.1% - 19.3% for the conical, and 25.7% - 46.9% for the Gaussian distribution. These limits, which are indicated by the horizontal lines in Fig. 5, agree with the results of the simulations, since the obtained "initial" emittance growth factors lie between these values.
We have performed in addition a series of computer simulations for $\sigma_0=60°$ over the range from $\sigma=60°$ to $\sigma=10°$ for both the K-V and the Gaussian distribution. Fig. 6 shows the r.m.s. emittance growth values after 50 cells. For the K-V distribution, no emittance growth occurs above $\sigma=10°$. Below 10°, a small growth that is due to a fourth-order instability mode[1] is obtained.
The emittance growth factors for the Gaussian distribution obtained from particle simulations lie between those calculated from the emittance growth formulae. This suggests that the Gaussian distribution, like the K-V distribution, is also stable with respect to fourth and higher-order modes under these conditions, and that the emittance growth can be explained in terms of the "homogenization" effect. This explanation is confirmed by the fact, that in all simulation runs at values of σ greater than 10° no further emittance growth is observed after the initial change of the charge distribution in the first period of the channel.

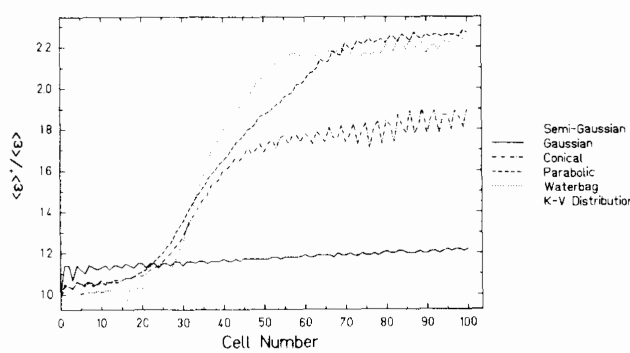

Fig. 3 Emittance growth factors versus the number of cells for different initial phase space distributions at $\sigma_0=90°$, $\sigma=41°$

3. Quasi-stationary Waterbag Distribution for Periodic Focusing

For continuous focusing, any type of phase space distribution is strictly self-consistent, if the isohamiltonians apply to a line of constant density. Since in the case of a "waterbag" distribution the filling of the phase space is uniform, only the boundary is important. For rotational symmetry, this boundary can be expressed as follows:

$$r'^2 + \frac{I_0(ka\cdot r) - 1}{I_0(ka) - 1} \le 1 \quad ,$$

where $r,r'\le 1$ are the normalized transverse variables. I_0 is the modified Bessel function and "ka" is a dimensionless intensity parameter, which is equal to zero for zero current. For that case the phase space boundary becomes:

$$r'^2 + r^2 \le 1 \quad ,$$

as it is for the conventional "waterbag" distribution. If we transform the latter case under space charge conditions in a continuous solenoid, we obtain aberrations as

shown in Fig. 7, whereas Fig. 8 shows the identical mapping of the "waterbag" distribution with the space charge matched phase space boundary.
If we perform a canonical transformation of these types of waterbag distributions matching the second moments to a quadrupole channel, we obtain the same results: the ellipsoidal phase space boundary causes space charge aberrations and emittance growth (which can be estimated with the emittance growth formulae), whereas the space charge matched waterbag type remains essentially unchanged, yielding no growth of the r.m.s. emittance, as shown in Fig. 9.

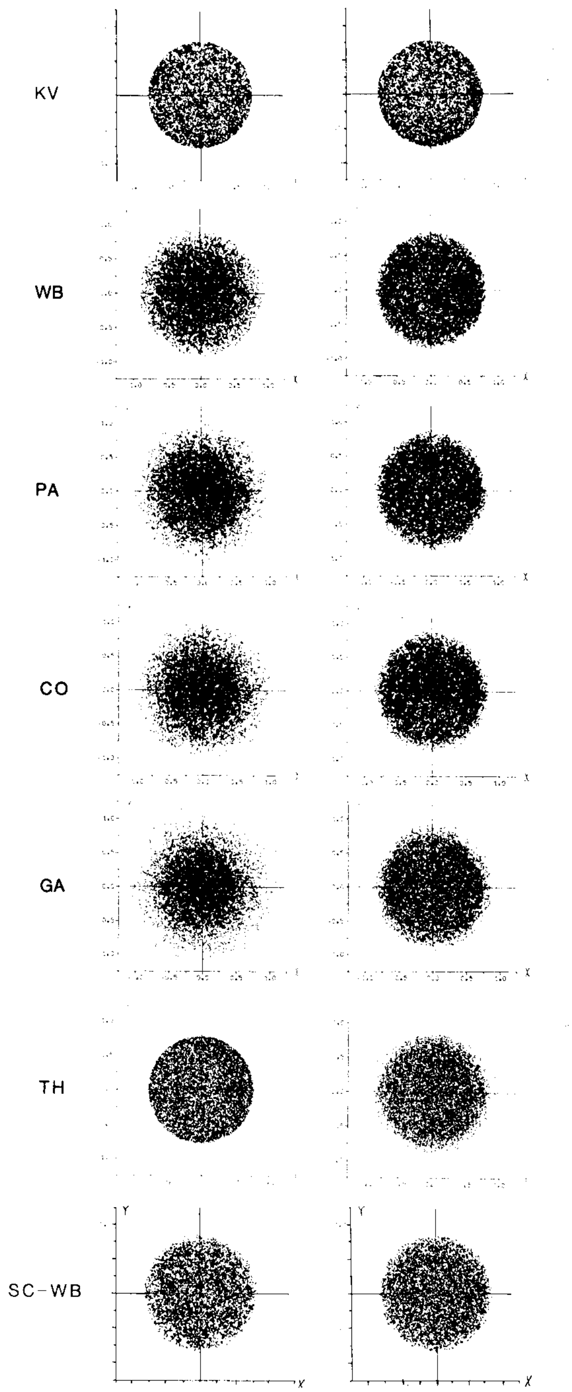

Fig. 4 Real space projections (x,y) of different initial phase space distributions before and after the first cell of the GSI FODO channel at $\sigma_0=60°$, $\sigma=15°$

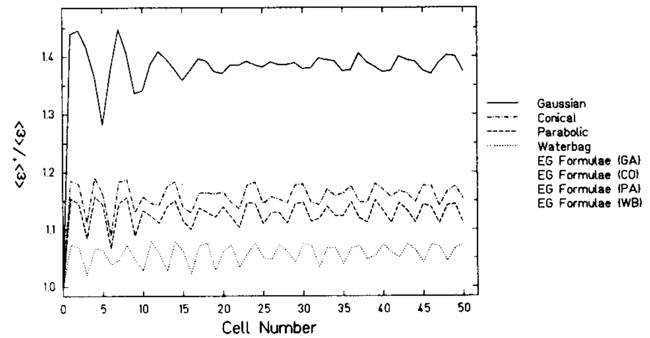

Fig. 5 Emittance growth factors versus the number of cells for different initial phase space distributions at $\sigma_0=60°$, $\sigma=15°$

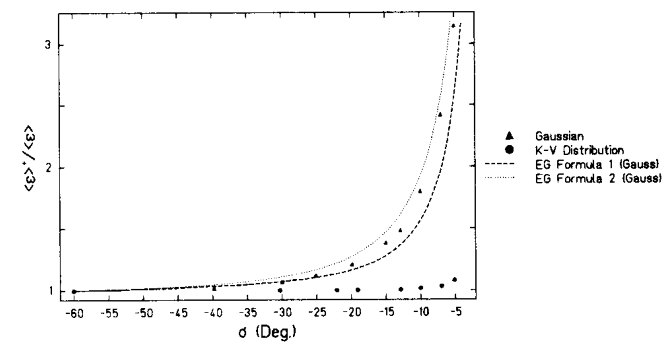

Fig. 6 Emittance growth factors after 50 cells versus σ for initial K-V- and Gaussian distributions at $\sigma_0=60°$

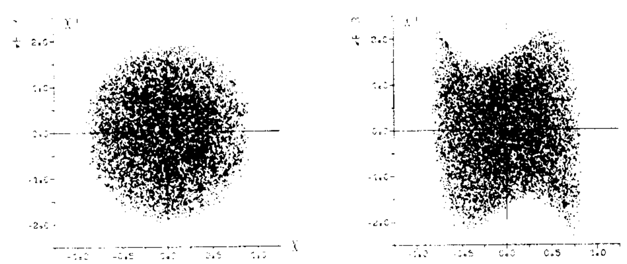

Fig. 7 r.m.s.-matched waterbag distribution with elliptical phase space boundary before and after the transformation through a continuous solenoid at $\sigma_0=60°$, $\sigma=15°$

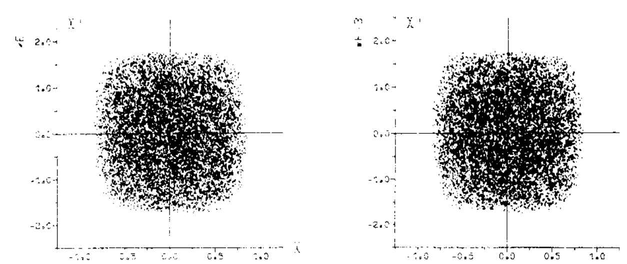

Fig. 8 Same as Fig. 7, but with space charge matched phase space boundary

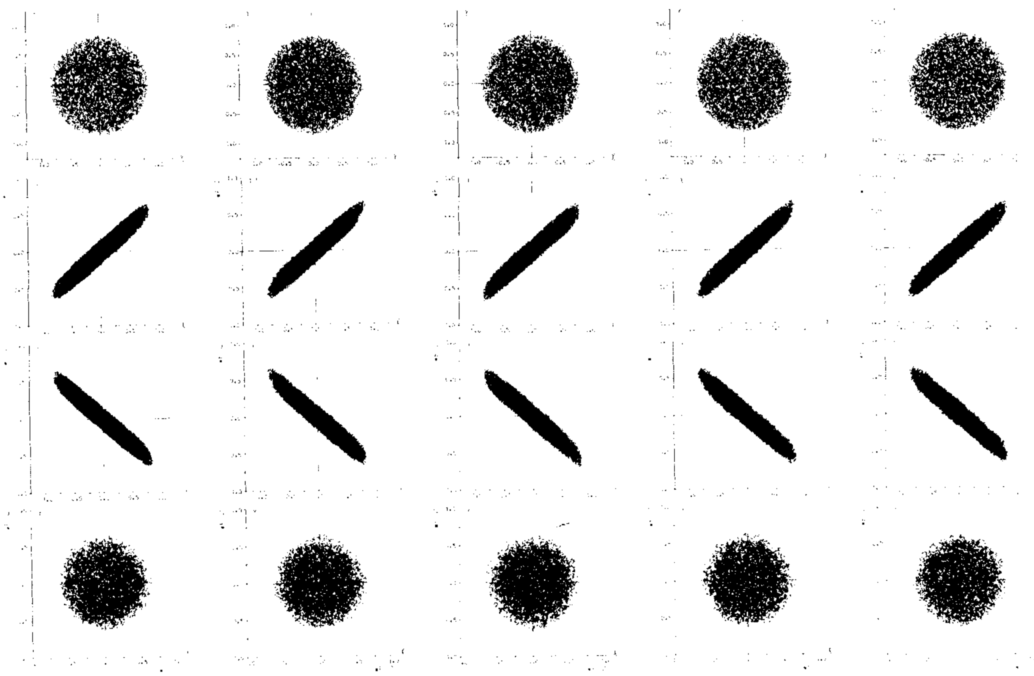

Fig. 9 Transformation of an initial r.m.s.-matched waterbag distribution with space charge matched, non-elliptical phase space boundary through the GSI FODO channel at $\sigma_0=60°$, $\sigma=15°$

4. Conclusions

Using the more realistic non-K-V phase space distributions for the simulation of beam transport, we have to distinguish two effects:

a) Resonances, that are very pronounced for the K-V distribution, are more or less damped for the others depending on the spread of tune σ inside the beam.
b) Redistributions of the phase space points occur due to an imbalance of internal and external forces.

This second effect occurs if we transform elliptically symmetric phase space distributions, which are conserved in the zero current limit due to the absence of non-linear forces, under space charge conditions. In that case, one obtains a homogenization of the charges in real space, associated with a growth of the transverse r.m.s. emittance. The upper and lower limit values can be estimated by the emittance growth formulae.

On the other hand, if we transform a Semi-Gaussian distribution in the low current region, we have a dehomogenization of the charge density in real space. This is accompanyed by a small shrinking of the r.m.s. emittance, since this type of distribution is only conserved in the infinite current limit.
Both cases can be seen as a change of the distribution towards a more self-consistent one.
It has been shown for the "waterbag" distribution that the latter effects can be avoided using a space charge matched phase space boundary that can be expressed analytically for axially symmetric continuous focusing. If a subsequent canonical transformation is applied matching the second moments to those required for periodic focusing, we obtain at least a quasi-stationary beam transport of a non-K-V distribution for non-continuous focusing.

References

[1] I. Hofmann, L.J. Laslett, L. Smith, and I.Haber, Part. Acc. **13**,145 (1983)
[2] J. Klabunde, M. Reiser, A. Schönlein, P. Spädtke, J. Struckmeier, IEEE Trans.Nucl.Sci. **30**,2543 (1983)
[3] J. Struckmeier, J. Klabunde, M. Reiser, GSI-Preprint 83-33; Part. Acc. **15**,47 (1984)