14th International Particle Accelerator Conference,Venice, Italy JACoW Publishing
ISBN: 978-3-95450-231-8 ISSN: 2673-5490 doi: 10.18429/JACoW-IPAC2023-WEPL014

HIGHER-ORDER SPIN DEPOLARIZATION ANALYSIS

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Abstract

Current and historic tracking studies of the RHIC accelerator lattice find difficulty in explaining the transmission efficiency of spin polarization from the AGS extraction to RHIC storage energies. In this paper, we discuss mechanisms that result in resonant depolarizing behavior, beyond the usual intrinsic and imperfection resonance structures. In particular, the focus of this paper will be on higher-order resonances that become apparent in the presence of snakes. The set of conditions that identify higher-order spin-orbit resonances are $\nu = j_0 + \vec{j} \cdot \vec{Q}$ for integers $(j_0, \vec{j}) \in \mathbb{Z}^4$, where ν is the spin tune and $\vec{Q}$ contains the orbit tunes. Note that we do not use the closed-orbit spin tune ν_0 but rather the amplitude-dependent spin tune $\nu(J_x, J_y, J_z)$ that depends on the phase-space amplitudes. While Sibrian snakes keep ν_0 at $1/2$, the amplitude-dependent spin tune can deviate from $1/2$ and can cross resonances during acceleration.

INVARIANT FRAME FIELD

Dynamics of spin motion are described by the T-BMT equation. For magnetic fields it is:

$$\frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}t} = -\frac{q}{m\gamma}\left[(1+G\gamma)\boldsymbol{B}_\perp + (1+G)\boldsymbol{B}_\parallel\right] \times \boldsymbol{S}$$

In an accelerator, the (static) magnetic fields along the comoving coordinate system are periodic in space with the ring's circumference C: $B(x, s+C, y) = B(x, s, y)$. However, in general, the fields along a particle's trajectory are not periodic with the circumference, $B(x(s+C), s+C, y(s+C)) \neq B(x(s), s, y(s))$. Therefore, except for a particle travelling on the closed-orbit, the spin motion is not periodic with C. If the T-BMT precession vector is written using action-angle coordinates, it becomes evident that it is periodic in the orbital phases and in the azimuth θ around the ring. Floquet's theorem allows writing the solution as a periodic envelope with a sinusoidal variation [1]:

$$\boldsymbol{S}\left(\vec{J}, \vec{\phi}(s), s\right) = U_{\vec{J}}(s, \vec{\phi}) \exp(\mathcal{J}\nu\theta) U^T_{\vec{J},0}\boldsymbol{S}_0$$

where $\nu(\vec{J})$ is known as an *Amplitude-Dependent Spin Tune*, and $U_{\vec{J}}(s, \vec{\phi}) = [\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{n}]_{\vec{J}}$ is known as an *Invariant Frame Field*, henceforth referred to as an ADST and IFF. There are situations where an IFF does not exist, such as when the beam is sitting on an orbital resonance since orbital motion is unstable in that case. We restrict further discussion to situations where IFFs do exist. The inverse matrix of the IFF in above equation transports the initial spin coordinates $\boldsymbol{S}_0$ into a system in which the spin rotates around the vertical by $\nu\theta$, before the IFF transports the spin back into the accelerator's coordinate system. For polarization, an IFF plays the role similar to the beta functions.

The axis around which the spin rotates in the IFF is $\boldsymbol{n}$, known as the *Invariant Spin Field* (ISF). It approaches the closed-orbit 1-turn precession vector $\boldsymbol{n} \to \boldsymbol{n}_0$ as orbital amplitudes approach zero. The projection of a particle's spin vector onto the ISF, $\boldsymbol{S} \cdot \boldsymbol{n} \equiv J_S$, has been proved to be an adiabatic invariant of spin motion [2]. It is the change of this quantity that is responsible for beam depolarization.

SPIN DEPOLARIZATION

The projection of each particle's spin onto the ISF is given by J_S and is constant during particle motion, as long as the fields in the accelerator do not change. If the fields change slowly as compared to the spin precession, for example during acceleration, J_S is an adiabatic invariant, i.e., it generally changes very little. To analyze in how far J_S does change when a parameter τ of the accelerator changes slowly, we construct the vector:

$$\boldsymbol{\eta} \equiv \frac{1}{2}\left(\boldsymbol{u}_1 \times \partial_\tau \boldsymbol{u}_1 + \boldsymbol{u}_2 \times \partial_\tau \boldsymbol{u}_2 + \boldsymbol{n} \times \partial_\tau \boldsymbol{n}\right) \qquad (1)$$

which satisfies $\partial_\tau \boldsymbol{n} = \boldsymbol{\eta} \times \boldsymbol{n}$ and similarly for $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. A slow increase in energy can be describe by such a parameter τ.

While at fixed energy spins precess around $\boldsymbol{n}$, changing energy causes $\boldsymbol{n}$ to rotate around $\boldsymbol{\eta}$ at each point in space, pushing $\boldsymbol{n}$ away from the instantaneous spin direction, which changes J_S. This change of J_S is especially strong when the change of $\boldsymbol{\eta}$ along a particle's trajectory is in resonance with the spin precession around $\boldsymbol{n}$. For this reason, a frequency analysis of $\boldsymbol{\eta}$ can give insight to depolarizing resonances.

With this in mind, it is those components of $\boldsymbol{\eta}$ which are perpendicular to $\boldsymbol{n}$ that give rise to perturbations of J_S. One can associate with each resonance condition $\nu \equiv \kappa \equiv \vec{j} \cdot \vec{Q} \bmod 1$ a resonance strength [3] such that spins obey the Froissart-Stora formula while ramping across isolated higher-order resonances. The resonance strength ε_κ at a resonance condition κ can be defined in terms of Fourier components of η,

$$\lim_{\mathrm{N}\to\infty} \frac{1}{2\pi\mathrm{N}} \int_0^{2\pi\mathrm{N}} \boldsymbol{\eta} \cdot (\boldsymbol{u}_1 + i\boldsymbol{u}_2) e^{-i\kappa\theta}\, \mathrm{d}\theta \qquad (2)$$

The detailed relation of these coefficients to resonance strengths will be analyzed in detail in future work.

RESONANCE STRENGTH

We now manipulate the previous expression, Eq. (2), into an elegant form using only the periodicity conditions of the IFF. Specifically, while the integrand in Eq. (2) is evaluated

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along the trajectory of a particle, the non-exponential component $\boldsymbol{\eta} \cdot (\boldsymbol{u}_1 + i\boldsymbol{u}_2)$ exists independently of the beam and is periodic in azimuth and betatron phase space:

$$
\begin{aligned}
&\lim_{N\to\infty} \frac{1}{2\pi N} \int_0^{2\pi N} \boldsymbol{\eta} \cdot (\boldsymbol{u}_1 + i\boldsymbol{u}_2) e^{-i\kappa\theta} d\theta \\
&= \lim_{N\to\infty} \frac{1}{2\pi N} \int_0^{2\pi N} \left(\sum_{j\in\mathbb{Z}^4} \eta_j \, e^{i(j_0\theta + \vec{j}\cdot\vec{\phi})} \right) e^{-i\kappa\theta} d\theta \\
&= \sum_{j\in\mathbb{Z}^4} \eta_j \, \delta\left(j_0 + \vec{j}\cdot\vec{Q} - \kappa\right) \; \frac{1}{2\pi} \int_0^{2\pi} e^{i\vec{j}\cdot\left(\vec{\phi}(\theta) - \vec{Q}\theta\right)} d\theta \\
&= \sum_{j\in\mathbb{Z}^4} \eta_j \left\langle e^{i\vec{j}\cdot\left(\vec{\phi}(\theta)-\vec{Q}\theta\right)} \right\rangle \delta\left(j_0 + \vec{j}\cdot\vec{Q} - \kappa\right)
\end{aligned}
$$

In the last line, we express the 1-turn integral as an average $\frac{1}{2\pi}\int_0^{2\pi} ... \equiv \langle ... \rangle$, since the exponential term is periodic in θ. The Fourier coefficients η_j are independent of any particle trajectories, and are defined as an integral over the 4-torus $\phi^\mu \equiv (\theta, \vec{\phi}) \in T^4$:

$$
\eta_j \equiv \frac{1}{(2\pi)^4} \int_{T^4} \boldsymbol{\eta} \cdot (\boldsymbol{u}_1 + i\boldsymbol{u}_2) e^{-ij_\mu \phi^\mu} d^4\phi
$$

This expression for resonance strength ε_κ that enters the Froissart Stora formula is useful for analyzing spin dynamics and for optimizing accelerator lattices for minimal polarization loss during acceleration. Since Fourier terms of $\boldsymbol{\eta}$ are related to the higher-order resonance strength, they can be used for accelerator optimization. The IFF is usually attained through particle tracking, e.g., by stroboscopic averaging. However, methods that do not require tracking also exist, e.g., Differential Algebra normal form theory of spin-orbit motion can also compute $\boldsymbol{\eta}$ without the need for tracking.

This nonlinear resonance strength ε_κ has several features that are known from 1st-order resonances: they describe Froissart-Stora polarization loss while accelerating through a resonance, and they describe spin tune jumps by $2\varepsilon_\kappa$ at the location of a resonance. This observation yields a method for computing the resonance strength that is alternative to analyzing $\boldsymbol{\eta}$. One can compute the amplitude dependent spin tune as a function of energy and observe by how far it jumps when ν crosses a resonance condition, this jump is $2\varepsilon_\kappa$. Furthermore, one observes how the amplitude dependent spin tune changes with energy during the ramp. This gives the rate of spin-tune change α. The Froissart-Stora formula depends on ε_κ and α and describes depolarization during crossing of higher-order resonances.

EXAMPLE

In the following section, we find an example higher-order resonance in RHIC by looking in the region of strong first-order intrinsic resonances *without* Siberian snakes. Using well-established perturbative matrix methods [4], we calculate the resonance spectrum for a particle with normalized amplitude of 10π mm · mrad, as shown in Fig. 1.

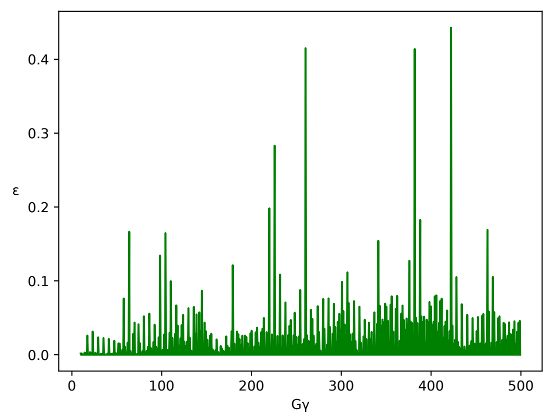

Figure 1: Resonance strength ε_{ν_0} for equivalence classes $\nu_0 = G\gamma \equiv \pm Q_y$ mod1 and vertical emittance $J_y = 10\pi$ µm.

Turning on the effect of ideal Siberian snakes fixes the closed-orbit spin tune at $\nu_0 = 1/2$, which prevents the crossing of any 1st-order intrinsic resonances. Nevertheless, higher-order intrinsic resonances can be crossed by large-amplitude particles for which the spin tune deviates from $1/2$ [5]. Polarization tends to be reduced in two energy regions during the ramps of RHIC. These are energy regions where strong 1st-order resonances would be crossed without Siberian snakes.

Upon focusing around the vicinity of the second strongest of these regions and increasing the vertical emittance, we find strong dips in the equilibrium polarization of the ISF. As seen in Fig. 2, associated with these dips are spin tune jumps whose size determines twice the resonance strength (at that orbital amplitude).

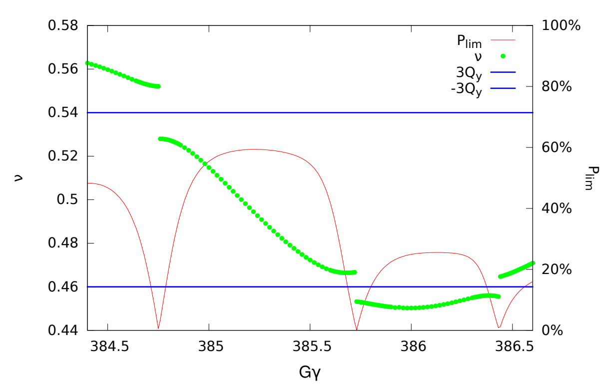

Figure 2: Jumps in the ADST (ν) in RHIC for a particle with 40 µm vertical emittance.

While in other regions the ADST is a smooth function of energy, here several jumps can be observed, showing that nonlinear depolarizing resonances are crossed, even though the closed orbit spin tune ν_0 remains $1/2$ at all times. Resonance conditions are indicated by horizontal lines, and it is evident that the spin tune jumps symmetrically across resonance lines.

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Finally, in Figs. 3 and 4, we present the ISF at the azimuth of IR6 overlayed on the unit sphere as a function of vertical betatron phase $\boldsymbol{n}(\phi_y)$ at a non-resonant as well as a resonant energy. These plots showcase the lowering of equilibrium polarization as the beam approaches a generic higher-order resonance condition. It is interesting to note that due to the midplane symmetry of the RHIC lattice, one can see the symmetry of the ISF under reflection $\phi_y \to -\phi_y$ in these figures. This particular feature allows RHIC to avoid all even higher-order resonances.

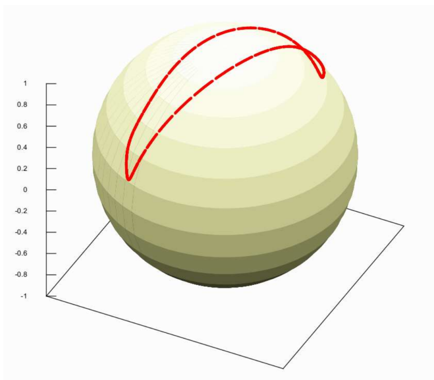

Figure 3: The ISF away from resonance at $G\gamma = 385.3$ and vertical emittance $J_y = 10\pi$ µm.

The fact that higher-order depolarizing resonances can be computed with the presented method may give a handle for advanced accelerator optimization. One now can optimize designs in order to reduce the critical ε_κ.

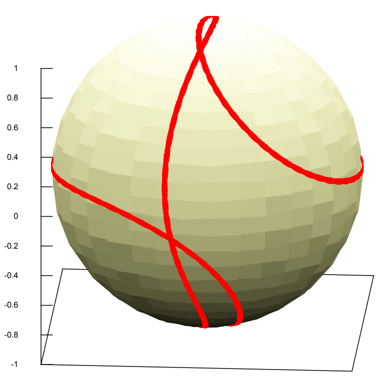

Figure 4: The ISF in the immediate vicinity of resonance at $G\gamma = 386.4$ and vertical emittance $J_y = 10\pi$ µm.

CONCLUSION

In this paper, we present a framework for identifying higher-order resonances and calculating resonance strength. Standard methods for calculation the invariants spin field typically depend on particle tracking, for example by means of stroboscopic averaging [2], but tracking-independent methods are also available, e.g., Differential Algebra normal form theory.

Computing the most critical higher-order spin-orbit resonance strength with Siberian snakes opens opportunities for advanced accelerator optimizations for the maximization of high-energy polarization.

ACKNOWLEDGEMENT

Many helpful discussions with Desmond Barber, Jim Ellison, and Klaus Heinemann are much appreciated. This work has been supported by Brookhaven Science Associates, LLC under Contract No. DESC0012704 with the U.S. Department of Energy.

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