Radiation of a charged particle bunch moving along a deep corrugated surface with a small period

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The equivalent boundary conditions

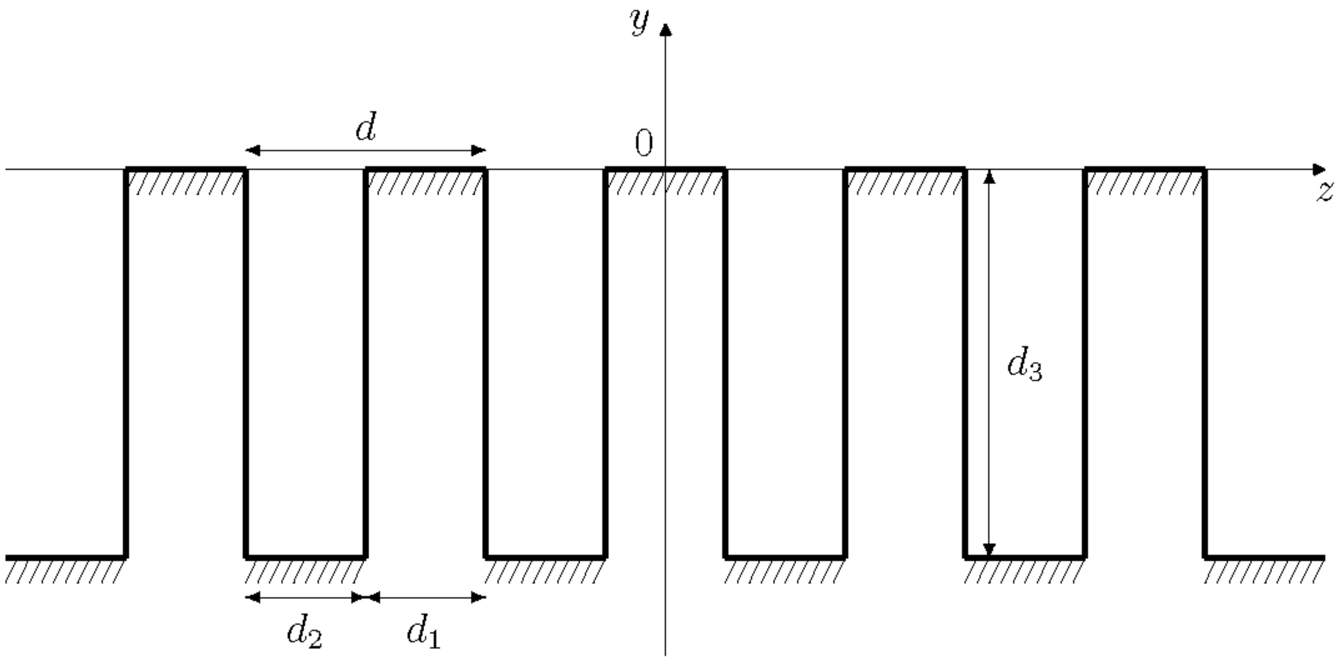

Fig.1. The corrugated structure.

The equivalent boundary conditions (EBC) can be used under condition [1]:

$$d << \lambda$$

The EBC [1]: $E_{\omega z} = \eta^m H_{\omega x}, \quad E_{\omega x} = 0$

The deep corrugation: $d << d_3$

The impedance [1]:

$$\eta^m = i\frac{d_2}{d}\frac{\mathrm{tg}(k_0 d_3)}{1 - k_0 d\, l\, \mathrm{tg}(k_0 d_3)}, \quad k_0 = \frac{\omega}{c}$$

Parameter l [1]: $l = \frac{1}{2\pi}\left[(2-\xi)^2 \ln(2-\xi) - \xi^2 \ln\xi - 2(1-\xi)\ln 4(1-\xi)\right], \quad \xi = \frac{d_1}{d}, \quad 0 < l << 1$

[1] E.I. Nefedov and A.N. Sivov, *Electrodynamics of periodic structures,* Nauka, Moscow, Russia, (1977)

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The problem definition

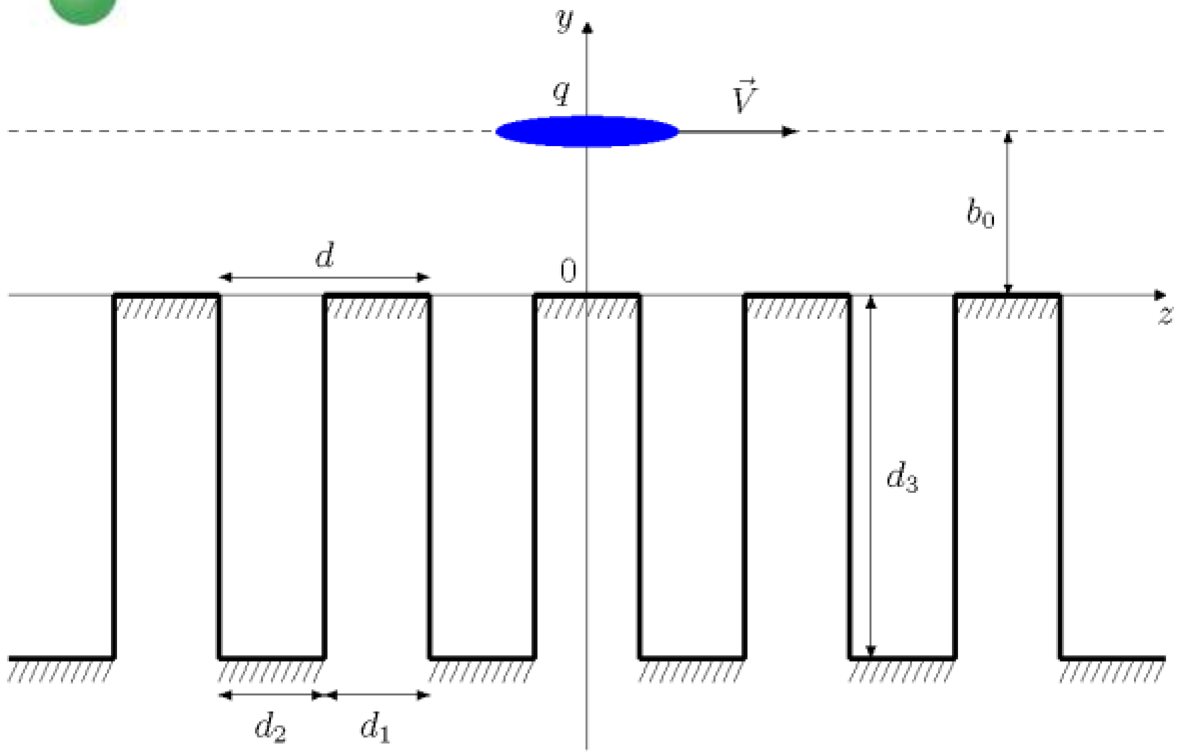

Fig.2. The corrugated structure and the moving bunch.

The velocity of the bunch:

$$\vec{V} = V_z \vec{e}_z, \;\; V_z \equiv V = c\beta$$

The charge and current densities:

$$\rho = q\delta(x)\delta(y - b_0)\kappa(z - Vt), \;\; j_z \equiv j = V\rho$$

The total field: $\vec{\Pi} = \vec{\Pi}^{(i)} + \vec{\Pi}^{(r)}$

$\vec{\Pi}^{(i)}$ - the «forced» field

$\vec{\Pi}^{(r)}$ - the «free» field

The Helmholtz equation for the Fourier-transforms of the Hertz potential: $\left\{\Delta + k_0^2\right\}\vec{\Pi}_\omega = -\dfrac{4\pi i}{ck_0}\vec{j}_\omega$

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The Hertz potential of the total field

The «forced» Hertz potential: $\vec{\Pi}^{(i)} = \Pi_z^{(i)} \vec{e}_z$

$$\Pi_{\omega z}^{(i)} = -\frac{q\tilde{\kappa}}{k_0 c} e^{i\frac{k_0 z}{\beta}} \int_{-\infty}^{+\infty} dk_x \frac{e^{ik_x x + ik_{y0}|y-b_0|}}{k_{y0}}, \quad k_{y0} = i\sqrt{k_x^2 + k_0^2 \frac{1-\beta^2}{\beta^2}}, \ \operatorname{Im} k_{y0} > 0$$

The Fourier-transform of the bunch profile: $\tilde{\kappa} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\zeta \, \kappa(\zeta) \exp\left(-i \frac{k_0}{\beta} \zeta\right), \ \zeta = z - Vt$

The «free» Hertz potential: $\vec{\Pi}^{(r)} = \Pi_x^{(r)} \vec{e}_x + \Pi_z^{(r)} \vec{e}_z$

$$\Pi_{\omega x}^{(r)} = -\frac{q\tilde{\kappa}}{k_0 c} e^{i\frac{k_0 z}{\beta}} \int_{-\infty}^{+\infty} dk_x R_x \frac{e^{ik_x x + ik_{y0}(y+b_0)}}{k_{y0}}, \quad \Pi_{\omega z}^{(r)} = -\frac{q\tilde{\kappa}}{k_0 c} e^{i\frac{k_0 z}{\beta}} \int_{-\infty}^{+\infty} dk_x R_z \frac{e^{ik_x x + ik_{y0}(y+b_0)}}{k_{y0}}$$

Relations between the Fourier-transforms of
the electromagnetic field and the Hertz potential: $\vec{E}_\omega = \vec{\nabla} \operatorname{div} \vec{\Pi}_\omega + k_0^2 \vec{\Pi}_\omega, \ \vec{H}_\omega = -ik_0 \operatorname{rot} \vec{\Pi}_\omega$

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The «free» field

The equivalent boundary conditions:
$$\begin{cases} E_{\omega z}^{(i)} + E_{\omega z}^{(r)} = \eta^m \left(H_{\omega x}^{(i)} + H_{\omega x}^{(r)} \right) \\ E_{\omega x}^{(i)} + E_{\omega x}^{(r)} = 0 \end{cases}$$

The expressions for the coefficients:

$$R_x = \frac{2k_0 k_x k_{y0} \beta \eta^m}{k_0^3 + \beta^2 \left(k_x^2 - k_0^2 \right)\left(k_0 - k_{y0}\eta^m \right)}, \quad R_z = -\frac{k_0^3 + \beta^2 \left(k_x^2 - k_0^2 \right)\left(k_0 + k_{y0}\eta^m \right)}{k_0^3 + \beta^2 \left(k_x^2 - k_0^2 \right)\left(k_0 - k_{y0}\eta^m \right)}$$

$$\begin{Bmatrix} \Pi_{\omega x}^{(r)} \\ \Pi_{\omega z}^{(r)} \end{Bmatrix} = -\frac{q\tilde{\kappa}}{k_0 c} e^{i\frac{k_0 z}{\beta}} \int_{-\infty}^{+\infty} dk_x \begin{Bmatrix} R_x \\ R_z \end{Bmatrix} \frac{e^{ik_x x + ik_{y0}\left(y + b_0 \right)}}{k_{y0}}, \quad k_{y0} = i\sqrt{k_x^2 + k_0^2 \frac{1-\beta^2}{\beta^2}}, \ \mathrm{Im}\, k_{y0} > 0$$

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The «free» field. The singularities of the integrands

$$\begin{Bmatrix} \Pi_{\omega x}^{(r)} \\ \Pi_{\omega z}^{(r)} \end{Bmatrix} = -\frac{q\tilde{\kappa}}{k_0 c} e^{i\frac{k_0 z}{\beta}} \int_{-\infty}^{+\infty} dk_x \begin{Bmatrix} R_x \\ R_z \end{Bmatrix} \frac{e^{ik_x x + ik_{y0}(y+b_0)}}{k_{y0}}$$

The branch points:

$$k_{y0} = i\sqrt{k_x^2 + k_0^2 \frac{1-\beta^2}{\beta^2}} = 0 \quad => \quad \pm k_{xb} = \pm i\frac{k_0}{\beta}\sqrt{1-\beta^2}$$

The poles are found from the dispersion equation:

$$k_0^3 + \beta^2\left(k_x^2 - k_0^2\right)\left(k_0 - ik_{y0}\eta_0^m\right) = 0, \quad \eta_0^m = \operatorname{Im}\eta^m$$

The solutions:

$$\pm k_{x0} = \pm k_0\sqrt{1 - \frac{\operatorname{sgn}\left(\eta_0^m\right)}{2\eta_0^{m2}}\left(\sqrt{1 + \frac{4\eta_0^{m2}}{\beta^2}} - \operatorname{sgn}\left(\eta_0^m\right)\right)}$$

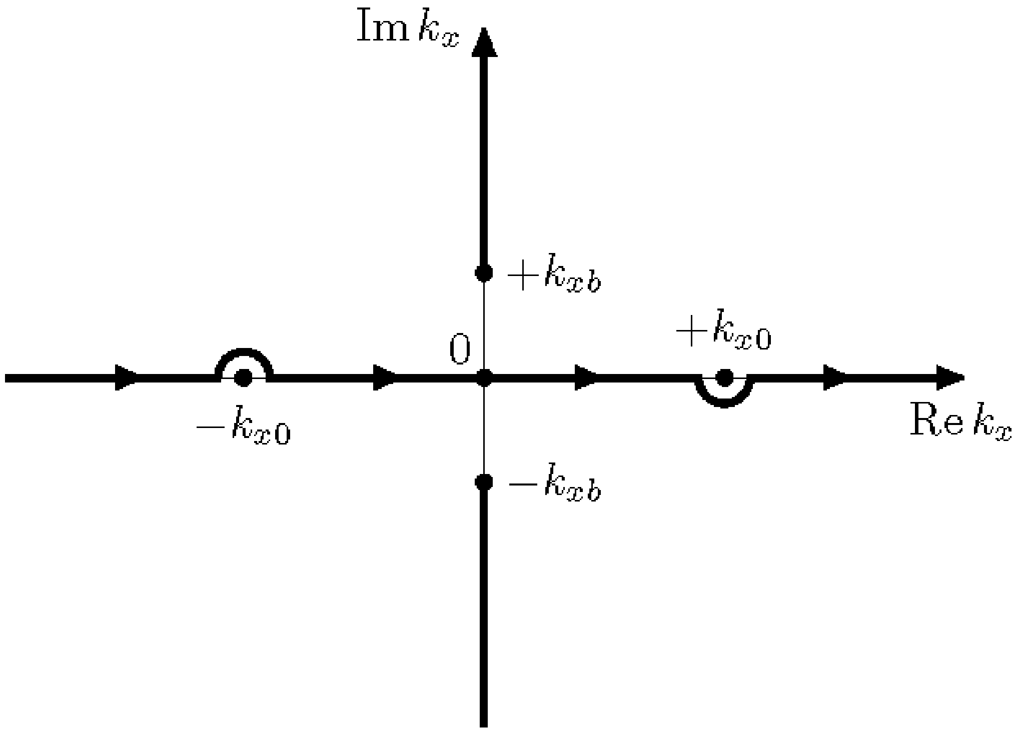

Fig.3. The complex plane of k_x: the integration path (real axis), poles $\pm k_{x0}$, branch points $\pm k_{xb}$ and the cuts.

The poles are real if $\beta > \dfrac{1}{\sqrt{1+\eta_0^{m2}}}$

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The «free» field. The saddle point method

The new integration variable χ: $k_x = \frac{k_0}{\beta}\sqrt{1-\beta^2}\,\mathrm{sh}\chi$

The phase: $\Phi(\chi) = \frac{k_0}{\beta}\sqrt{1-\beta^2}\left[x\,\mathrm{sh}\chi + i(y+b_0)\,\mathrm{ch}\chi\right]$

The saddle point: $d\Phi(\chi)/d\chi = 0 \quad \Rightarrow \quad \chi_s = i\chi_s''$

The steepest descent path are found from the system: $\begin{cases} \mathrm{Re}\,\Phi(\chi) = \mathrm{Re}\,\Phi(\chi_s) \\ \mathrm{Im}\,\Phi(\chi) > \mathrm{Im}\,\Phi(\chi_s) \end{cases}$

The estimation of the integrals over the steepest descent path:

$$\Pi^{(r)}_{\omega x,z} \sim \frac{\exp\left[-k_0\beta^{-1}\sqrt{1-\beta^2}\sqrt{x^2+(y+b_0)^2}\right]}{\left[x^2+(y+b_0)^2\right]^{1/4}}$$

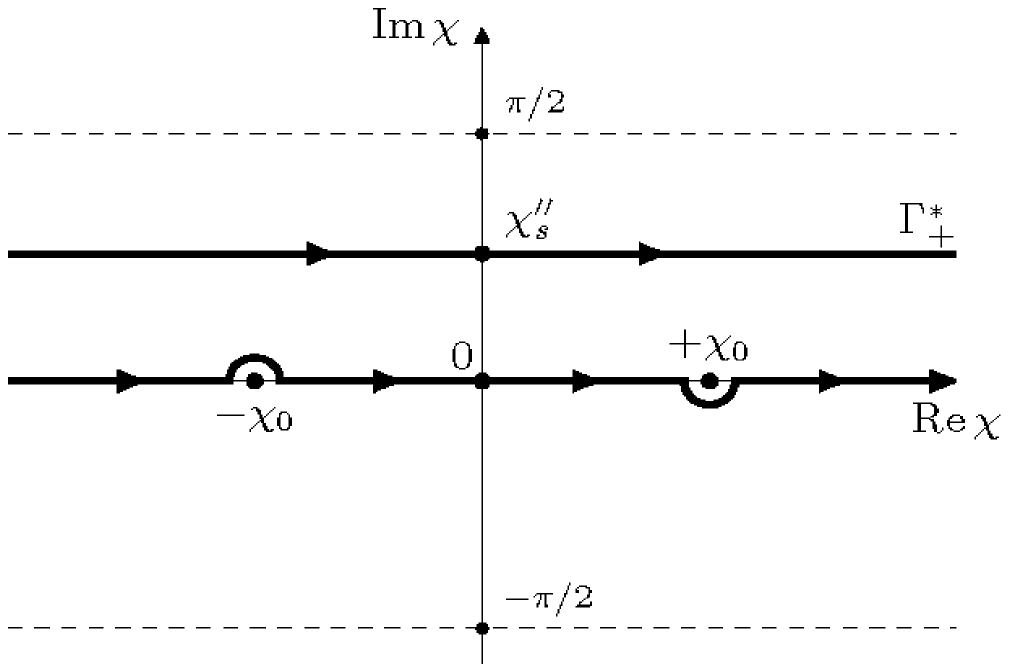

Fig.4. The complex plane of χ: the initial integration path, poles $\pm\chi_0$, saddle point χ_s and the steepest descent path Γ_+^* (for $x>0$).

We can neglect the contribution of the saddle point under condition:

$$k_0\beta^{-1}\sqrt{1-\beta^2}\sqrt{x^2+(y+b_0)^2} \gg 1$$

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The surface waves

The contributions of the poles: $\Pi^{(\mathrm{r})}_{\omega x,z} \approx \Pi^{(\mathrm{s})}_{\omega x,z} = 2\pi i\,\mathrm{sgn}(x) \operatorname*{Res}_{\pm k_{x0}} \Pi^{(\mathrm{r})}_{\omega x,z}$

The Fourier-transforms of the electromagnetic field of the surface wave:

$$E^{(s)}_{\omega x} = 0 \qquad H^{(s)}_{\omega x} = \frac{2\pi i q\tilde{\kappa}}{c} \frac{k_0 |\eta_0^m| g^2}{\beta\sqrt{\beta^2 - \mathrm{sgn}(\eta_0^m) g^2}} e^{\Phi(k_0,\vec{R})}$$

$$E^{(s)}_{\omega y} = -\frac{2\pi i q\tilde{\kappa}}{c} \frac{k_0 \eta_0^m}{\sqrt{\beta^2 - \mathrm{sgn}(\eta_0^m) g^2}} e^{\Phi(k_0,\vec{R})} \qquad H^{(s)}_{\omega y} = -\mathrm{sgn}(x) \frac{2\pi q\tilde{\kappa}}{c} \frac{k_0 (g^2 - \mathrm{sgn}(\eta_0^m))}{g^2} e^{\Phi(k_0,\vec{R})}$$

$$E^{(s)}_{\omega z} = \frac{2\pi q\tilde{\kappa}}{c} \frac{k_0 \beta (g^2 - \mathrm{sgn}(\eta_0^m))}{g^2\sqrt{\beta^2 - \mathrm{sgn}(\eta_0^m) g^2}} e^{\Phi(k_0,\vec{R})} \qquad H^{(s)}_{\omega z} = -\mathrm{sgn}(x) \frac{2\pi i q\tilde{\kappa}}{c} \frac{k_0 \eta_0^m}{\beta} e^{\Phi(k_0,\vec{R})}$$

$$\Phi(k_0,\vec{R}) = i\frac{k_0}{\beta}\sqrt{\beta^2 - \mathrm{sgn}(\eta_0^m) g^2}\,|x| + i\frac{k_0}{\beta} z - \frac{k_0 |\eta_0^m|}{\beta^2} g^2 (y + b_0), \quad g^2 = \frac{\beta^2}{2\eta_0^{m2}} \left(\sqrt{1 + 4\eta_0^{m2}\beta^{-2}} - \mathrm{sgn}(\eta_0^m)\right)$$

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The surface waves

The Fourier integrals: $E^{(s)}_{x,y,z} = \int\limits_{-\infty}^{+\infty} dk_0 E^{(s)}_{\omega x,y,z} e^{-k_0 ct}, \; H^{(s)}_{x,y,z} = \int\limits_{-\infty}^{+\infty} dk_0 H^{(s)}_{\omega x,y,z} e^{-k_0 ct}$

The Fourier-transform of the bunch profile:

$$\tilde{\kappa} = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} d\zeta \kappa(\zeta) \exp\left(-i \frac{k_0}{\beta} \zeta\right), \; \zeta = z - Vt$$

The Gaussian bunch:

$$\kappa_{gaus}(\zeta) = \frac{e^{-\zeta^2/2\sigma^2}}{\sqrt{2\pi}\sigma}, \quad \tilde{\kappa}_{gaus} = \frac{e^{-k_0^2\sigma^2/2\beta^2}}{2\pi}, \quad 2\sigma \text{ - the bunch length}$$

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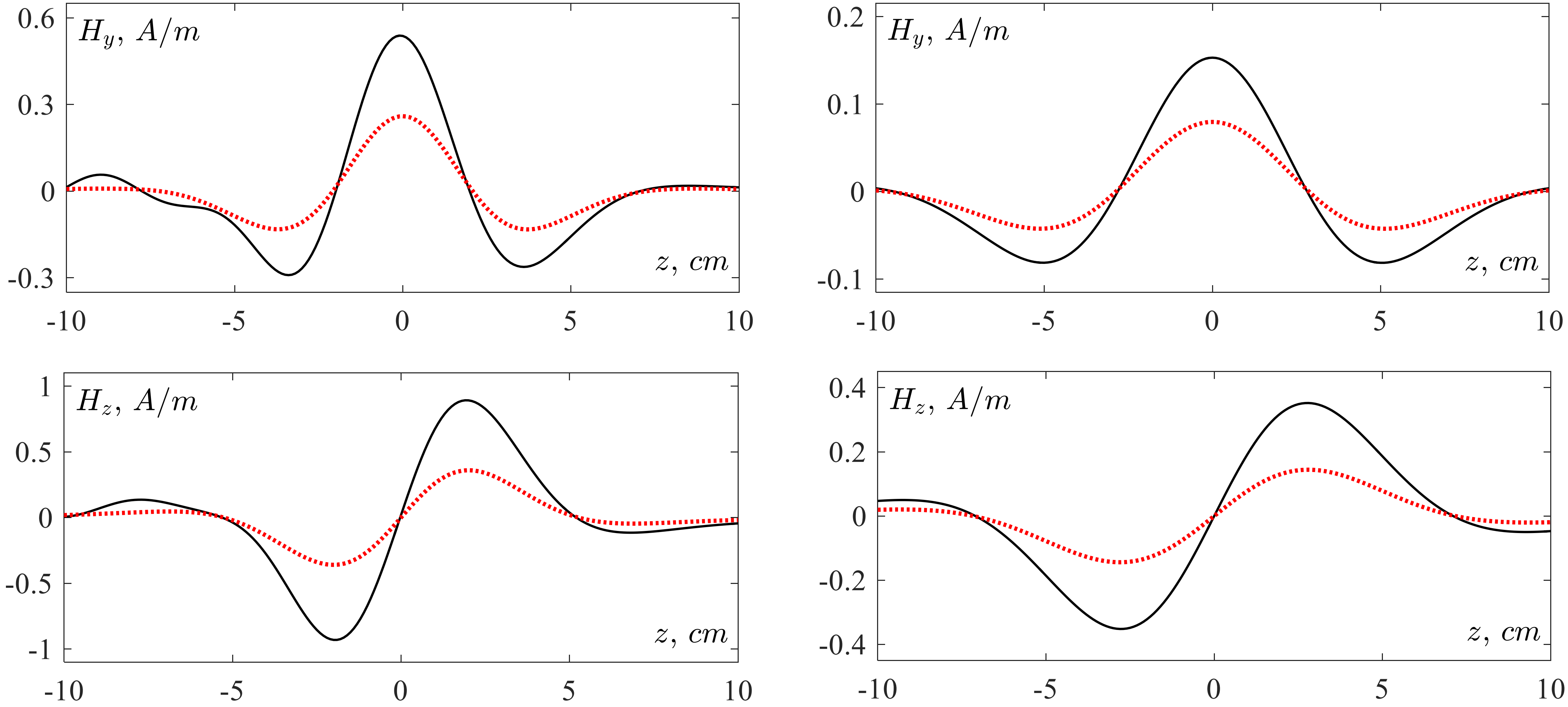

Fig.5. The components of the surface wave $H_y^{(s)}$ (top row) and $H_z^{(s)}$ (bottom row) depending on coordinate z for the Gaussian bunch with $q = 1\,\mathrm{nC}$. The bunch velocity is $\beta = 1$ (solid black curves) and $\beta = 0.75$ (dotted red curves). The bunch length is $2\sigma = 4\,\mathrm{cm}$ (left coloumn) and $2\sigma = 6\,\mathrm{cm}$ (right coloumn). The parameters: $d = 0.05\,\mathrm{cm}$, $d_1 = 0.01\,\mathrm{cm}$, $d_3 = 1\,\mathrm{cm}$, $b_0 = 2\,\mathrm{cm}$, $x = y = 0$, $t = 0$.

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The energy losses per the unit of the path length:

$$\frac{dW}{dz_0} = \frac{2}{c\beta}\int_{-\infty}^{+\infty} dz \int_0^{+\infty} dy\, S_x\Big|_{x>0}, \quad S_x = \frac{c}{4\pi}\left(E_y^{(s)}H_z^{(s)} - E_z^{(s)}H_y^{(s)}\right)$$

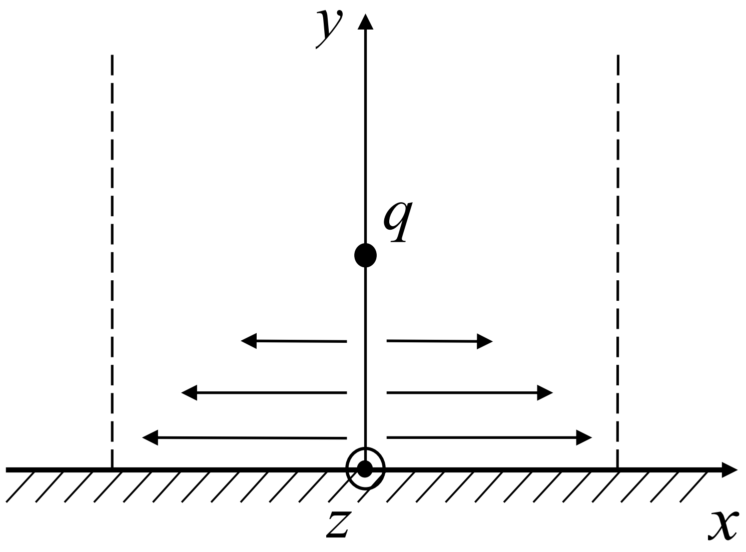

Fig.6. The energy flow of the charge.

The energy losses in terms of the Fourier-transforms:

$$\frac{dW}{dz_0} = 2c^2\int_0^{+\infty} dk_0 \int_0^{+\infty} dy\, \mathrm{Re}\left(E_{\omega y}^{(s)}H_{\omega z}^{(s)*} - E_{\omega z}^{(s)}H_{\omega y}^{(s)*}\right)$$

After the integration over y:

$$\frac{dW}{dz_0} = 4\pi^2 q^2 \beta \int_0^{+\infty} dk_0 k_0 \,|\tilde{\kappa}|^2 \frac{\eta_0^{m2} + \beta^2 g^{-4}(g^2 - \mathrm{sgn}(\eta_0^m))^2}{|\eta_0^m|\, g^2 \sqrt{\beta^2 - \mathrm{sgn}(\eta_0^m) g^2}} e^{-2k_0\beta^{-2}|\eta_0^m| g^2 b_0}$$

$$g^2 = \frac{\beta^2}{2\eta_0^{m2}}\left(\sqrt{1+4\eta_0^{m2}\beta^{-2}} - \mathrm{sgn}(\eta_0^m)\right), \quad \eta_0^m = \frac{d_2}{d}\frac{\mathrm{tg}\left(k_0 d_3\right)}{1 - k_0 d\, l\, \mathrm{tg}\left(k_0 d_3\right)}, \quad k_0 = \frac{\omega}{c}$$

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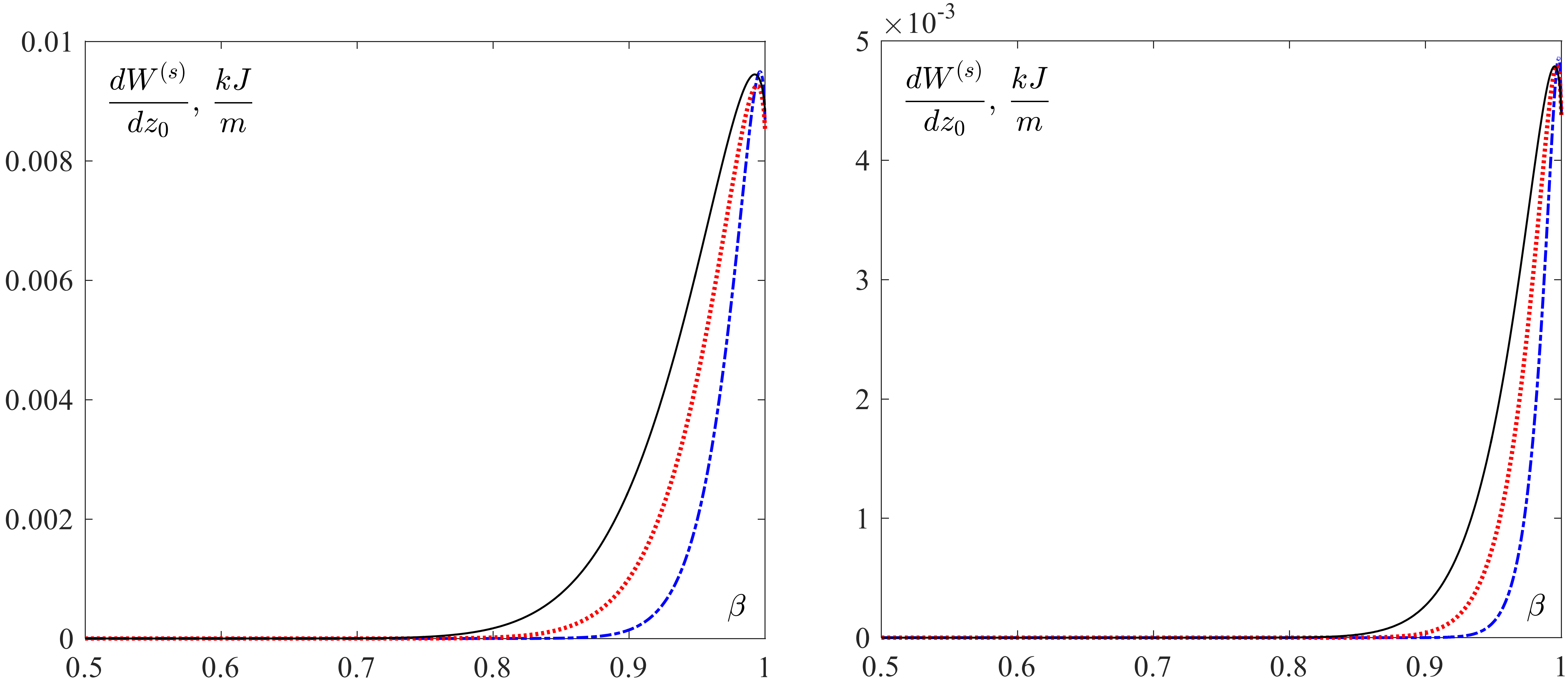

Fig.7. The energy of the surface wave $dW^{(s)}/dz_0$ depending on bunch velocity β for the Gaussian bunch with $q = 1\text{nC}$. The bunch length is $2\sigma = 4\,\text{cm}$ (left coloumn) and $2\sigma = 6\,\text{cm}$ (right coloumn). The depth of the structure is $d_3 = 1\,\text{cm}$ (solid black curves), $d_3 = 0.9\,\text{cm}$ (dotted red curves) and $d_3 = 0.8\,\text{cm}$ (dashed-dotted blue curves). The parameters: $d = 0.05\,\text{cm}$, $d_1 = 0.01\,\text{cm}$, $b_0 = 2\,\text{cm}$.

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Conclusions

- Investigated the relatively “longwave” radiation from the charged particle bunch moving along the corrugated conductive structure
- Studied the case of the deep corrugation when the depth of the structure is much greater than its period
- Obtained the general solution of the problem with the use of the equivalent boundary conditions
- Performed the asymptotic analysis of the general solution
- Shown that the volume radiation is absent but the surface waves can be generated at the frequencies under consideration
- Obtained the Fourier-transforms for the electromagnetic field of the surface wave
- Presented the results of numerical calculating the electromagnetic field of the surface wave
- Analyzed the energy of the surface radiation and obtained the formula for the energy per the unit of the path length
- Presented the dependences of the energy on the bunch velocity and the depth of the structure

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Thank you for attention!

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