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MODEL OF CURVATURE EFFECTS ASSOCIATED WITH SPACE CHARGE FOR LONG BEAMS IN DIPOLES*

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Abstract

For modeling the dynamics within a dipole of a bunch whose length is much larger than the vacuum pipe radius, it is typical to use a 2D (or 2.5D) Poisson solver, with arc length taken as the independent variable. However, sampled at a fixed time, the beam is curved, space charge is not truly 2D, and the usual cancellation between E and B contributions to the Lorentz force need not exactly hold. The size of these effects is estimated using an idealized model of a uniform torus of charge rotating inside a toroidal conducting pipe. Simple expressions are provided for the correction of the electric and magnetic fields to first order in the reciprocal of the curvature radius.

INTRODUCTION

A high-intensity beam bending in a dipole can be accurately modeled by tracking in time, using a 3D electromagnetic particle-in-cell solver with meshing of a (possibly) curved vacuum chamber. However, when the beam is long relative to the vacuum chamber radius, it is computationally more efficient to track using arc length as the independent variable, treating space charge with a 2D (or 2.5D) Poisson solver. This procedure neglects the curvature of the beam, under the assumption that 3D effects are largely shielded by the vacuum chamber.

To estimate the size of curvature effects, consider an unbunched beam bending in a uniform magnetic field B_{bend}. Particles at the design momentum $p_0 = mc\beta_0\gamma_0 = qB_{bend}R$ move on a circular orbit of radius R, which passes through the center of a toroidal pipe of radius a (Fig. 1). In the absence of collective effects, we treat the beam as a rotating torus of charge with cross-sectional charge density ρ. In this model, a particle at distance d from the axis of rotation moves with momentum $p = qB_{bend}d = p_0 d/R$. We compare the electric and magnetic fields of this toroidal system with those produced by a beam of cross-sectional density ρ in a straight cylindrical pipe of radius a.

Using 3D toroidal coordinates, the Poisson equation can be solved exactly in some cases [1,2]. However, the reference circle used to define standard toroidal coordinates does not coincide with the center of the conducting pipe, and the coordinates are related in a singular way to the local Frenet-Serret coordinates within the beam. Instead, we use a perturbative scheme with a more natural set of coordinates [3].

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SOLUTION FOR POTENTIALS

Consider the orthogonal coordinate system (r, θ, ζ) defined in terms of Cartesian coordinates (x, y, z) by:

$$x = (R + r\cos\theta)\cos(\zeta/R), \quad (1a)$$
$$y = r\sin\theta, \quad (1b)$$
$$z = (R + r\cos\theta)\sin(\zeta/R). \quad (1c)$$

Here, ζ measures arc length along a circular reference trajectory through the center of the toroidal pipe, R denotes the radius of the reference circle, and (r, θ) denote polar coordinates in the cross-section transverse to the toroidal pipe. See Fig. 1.

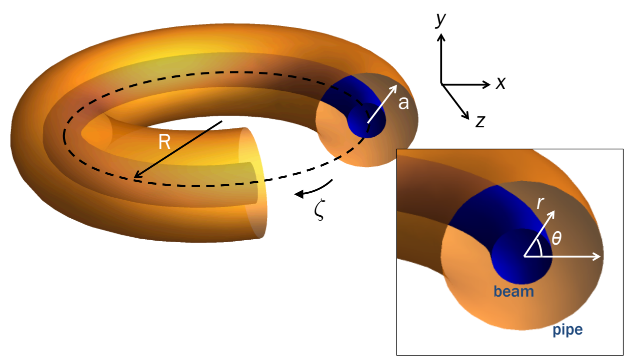

Figure 1: Unbunched beam in a toroidal conducting pipe, showing the coordinate system Eq. (1). The direction of beam current (inset) is out of the page. The bending is due to an applied uniform vertical magnetic field B_{bend}.

The metric in these coordinates is:

$$ds^2 = dr^2 + r^2 d\theta^2 + \left(1 + \frac{r\cos\theta}{R}\right)^2 d\zeta^2, \quad (2)$$

and the Laplacian of a function $f = f(r, \theta)$ is then:

$$\nabla^2 f = \nabla_\perp^2 f + \frac{1}{R + r\cos\theta}\left(\cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}\right), \quad (3)$$

where $\nabla_\perp$ is the usual 2D polar Laplacian in (r, θ).

Under the assumption that the system is static (no dependence on t), $\mathbf{E} = -\nabla\phi$ and $\mathbf{B} = \nabla\times\mathbf{A}$, and Maxwell's equations $\nabla\cdot\mathbf{E} = \rho/\epsilon_0$ and $\nabla\times\mathbf{B} = \mu_0\mathbf{J}$ can be expressed using potentials as:

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}, \qquad \nabla^2 A_\zeta - \frac{A_\zeta}{(R + r\cos\theta)^2} = -\mu_0 J_\zeta \quad (4)$$

with the boundary conditions $\phi(a, \theta) = 0$ and $A_\zeta(a, \theta) = 0$ at the conducting pipe $r = a$. Here we have assumed that the current density $\mathbf{J}$ points along the direction of motion.

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We may solve Eq. (4) systematically in powers of the small curvature parameter a/R. For example, expanding the potentials to linear order in a/R as:

$$\phi = \phi^{(0)} + \left(\frac{a}{R}\right)\phi^{(1)}, \qquad A_\zeta = A_\zeta^{(0)} + \left(\frac{a}{R}\right)A_\zeta^{(1)}, \quad (5)$$

and equating terms in Eq. (4) of like degree gives:

$$\nabla_\perp^2\phi^{(0)} = -\frac{\rho}{\epsilon_0}, \qquad \nabla_\perp^2 A_\zeta^{(0)} = -\frac{\beta_0}{c}\frac{\rho}{\epsilon_0}, \quad (6)$$

with the boundary conditions $\phi^{(0)}(a,\theta) = 0 = A_\zeta^{(0)}(a,\theta)$. As expected, Eq. (6) yields the potentials for a long beam with transverse density ρ propagating in a straight cylindrical pipe.

The first-order correction due to curvature is given by:

$$\nabla_\perp^2\phi^{(1)} = -\left(\frac{\cos\theta}{a}\right)\frac{\partial\phi^{(0)}}{\partial r} + \left(\frac{\sin\theta}{ar}\right)\frac{\partial\phi^{(0)}}{\partial\theta}, \quad (7a)$$

$$\nabla_\perp^2 A_\zeta^{(1)} = -\frac{\beta_0}{c}\frac{\rho}{\epsilon_0}\left(\frac{r\cos\theta}{\gamma_0^2 a}\right) + \frac{\beta_0}{c}\nabla_\perp^2\phi^{(1)}, \quad (7b)$$

with boundary conditions $\phi^{(1)}(a,\theta) = 0 = A_\zeta^{(1)}(a,\theta)$. The solutions of Eq. (7) can be obtained by using the Green's function for the 2D Poisson equation in a circular disk of radius a, namely:

$$G(r,\theta;r_0,\theta_0) = \frac{1}{4\pi}\ln\left\{\frac{a^2(r^2+r_0^2-2rr_0\cos(\theta-\theta_0))}{r^2r_0^2+a^4-2a^2rr_0\cos(\theta-\theta_0)}\right\}.$$

This follows since the unique solution u of the 2D problem:

$$\nabla_\perp^2 u = f, \qquad u(a,\theta) = 0 \quad (8)$$

is given by:

$$u(r,\theta) = \int_0^{2\pi}\int_0^a G(r,\theta;r_0,\theta_0)f(r_0,\theta_0)r_0dr_0d\theta_0. \quad (9)$$

MIDPLANE FIELDS

We evaluate the space charge fields in the case when the charge density ρ is uniform within a beam of radius $b < a$:

$$\rho(r,\theta) = \begin{cases}\rho_0, & 0 \le r \le b\\ 0, & b < r \le a\end{cases}. \quad (10)$$

Since the charge density and potentials are independent of the coordinate ζ, the electric field components are given by:

$$E_r = -\frac{\partial\phi}{\partial r}, \quad E_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}, \quad E_\zeta = 0, \quad (11)$$

and the magnetic field components are given by:

$$B_r = \frac{1}{r}\frac{\partial A_\zeta}{\partial\theta} - \left(\frac{\sin\theta}{R+r\cos\theta}\right)A_\zeta, \quad (12a)$$

$$B_\theta = -\frac{\partial A_\zeta}{\partial r} - \left(\frac{\cos\theta}{R+r\cos\theta}\right)A_\zeta, \quad (12b)$$

$$B_\zeta = 0. \quad (12c)$$

Our choice of boundary conditions for the potentials yields that $E_\theta(a,\theta) = 0$ and $B_r(a,\theta) = 0$, as required.

Expanding the fields in the form ($j = r, \theta$, or ζ):

$$E_j = E_j^{(0)} + \left(\frac{a}{R}\right)E_j^{(1)}, \qquad B_j = B_j^{(0)} + \left(\frac{a}{R}\right)B_j^{(1)}, \quad (13)$$

the fields in the absence of curvature ($a/R \to 0$) can be obtained by solving (6), yielding:

$$E_r^{(0)}(r,\theta) = \frac{\rho_0}{2\epsilon_0}\begin{cases}r, & 0 \le r \le b\\ b^2/r, & b < r \le a\end{cases}, \quad (14a)$$

$$B_\theta^{(0)}(r,\theta) = \frac{\beta_0}{c}E_r^{(0)}(r,\theta). \quad (14b)$$

All other field components vanish.

The correction to the fields to first order in a/R can be obtained by solving Eq. (7). The resulting integrals involving the Green's function G are difficult to evaluate in closed form. However, by differentiating the Green's function under the integral sign to obtain integrals for the field components Eqs. (11-12), and by considering only observation points within the plane of the bend, given by $\theta = 0$ or $\theta = \pi$, these integrals can be evaluated explicitly.

It is helpful to express the final results using local 2D Cartesian coordinates in the cross-section of the beam, defined by:

$$x = r\cos\theta, \qquad y = r\sin\theta, \quad (15)$$

so that the plane of the bend is defined by $y = 0$. Then we obtain for $-b \le x \le b$ (inside the beam):

$$E_x^{(1)}/E_{edge} = \frac{b}{8a}\left(\frac{b^2}{a^2} - \frac{3x^2}{b^2} - 4\ln\frac{b}{a}\right), \quad (16a)$$

$$B_y^{(1)}/B_{edge} = \frac{b}{8a}\left(\frac{3b^2}{a^2} + \frac{7x^2}{b^2} - 8 + 4\ln\frac{b}{a}\right),$$

while for $|x| > b$ (outside the beam):

$$E_x^{(1)}/E_{edge} = \frac{b}{8a}\left(\frac{b^2}{a^2} + \frac{b^2}{x^2} - 4 - 4\ln\frac{|x|}{a}\right), \quad (16b)$$

$$B_y^{(1)}/B_{edge} = \frac{b}{8a}\left(\frac{3b^2}{a^2} + \frac{3b^2}{x^2} - 4 + 4\ln\frac{|x|}{a}\right).$$

Here we have normalized the results by:

$$E_{edge} = \frac{\rho_0 b}{2\epsilon_0}, \qquad B_{edge} = \frac{\beta_0}{c}\frac{\rho_0 b}{2\epsilon_0}, \quad (17)$$

which denote the maximum values of the unperturbed fields Eq. (14), which occur at the beam edge $r = b$.

Figures 2-3 illustrate the behavior of Eq. (16) when the ratio of beam size to the pipe radius is $b/a = 1/5$. Within the beam, the unperturbed fields vary linearly with position x, while the first-order curvature correction varies quadratically with x. While the unperturbed fields vanish at the center of the beam ($x = 0$), this is not the case in the presence of curvature. It is straightforward to verify that particles at the center of the beam experience a net Lorentz force in the $x-$direction (directed away from the center of curvature).

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Note that this force depends on the momentum distribution of the beam, through its coupling with the magnetic field. For a realistic beam distribution, this may not be well-represented by this toy model.

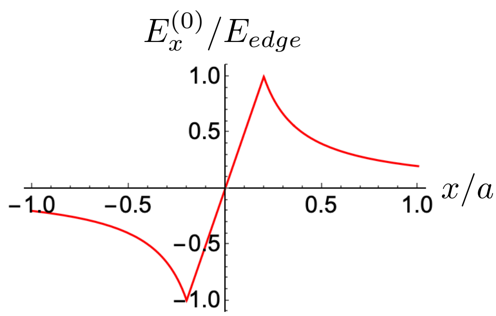

Figure 2: Space charge fields Eq. (14) for a long uniform beam in the zero-curvature limit for $b/a = 1/5$, shown in the bending plane in coordinates Eq. (15). The behavior of $B_y^{(0)}/B_{edge}$ is identical.

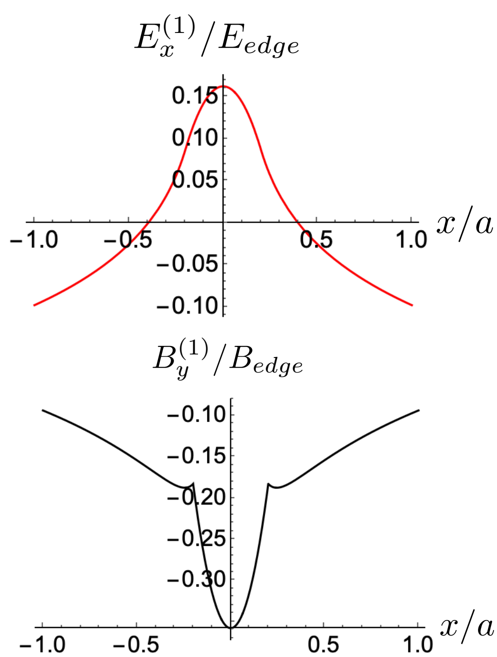

Figure 3: First-order curvature correction Eq. (16) to the space charge fields of a long uniform beam for the case $b/a = 1/5$. The beam edge is located at $x/a = \pm 1/5$.

In all cases, the first-order curvature correction to both fields is largest at the center of the beam. Figure 4 shows the electric and magnetic field values at the center of the beam as a function of the ratio b/a. Note that the relative correction to the magnetic field is larger than the relative correction to the electric field, over the full range of the ratio b/a. Since the corrections $E_x^{(1)}$ and $B_y^{(1)}$ are scaled by a/R in Eq. (13), these effects are small for typical values $a/R \sim 1/100$. For example, for a beam with $a = 2.5$ cm, $b = 2.5$ mm, and $R = 1$ m, this gives field corrections $\Delta E_x/E_{edge} = 0.3\%$ and $\Delta B_y/B_{edge} = 0.5\%$.

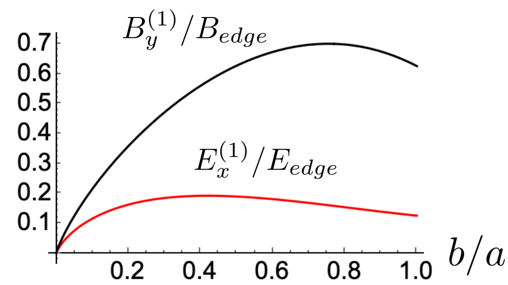

Figure 4: First-order curvature correction at the center of the beam as a function of the ratio b/a.

CONCLUSION

Using a simplified model of a long beam bending in a dipole, we have obtained simple expressions Eq. (16) relating the first-order curvature correction to the space charge fields with the physical parameters of interest: the beam size, the curvature radius, and the vacuum pipe radius. The effect of a conducting vacuum chamber with circular cross section is included. The purpose of this work is to estimate the regime of validity for space charge models based on a 2D Poisson solver in the presence of curvature, for unbunched beams. For a numerical method to compute the curvature correction for beams of finite bunch length in bends, see [3]. We have made no attempt to consider self-consistent dynamical effects resulting from curvature, such as emittance growth. For estimates of these effects for bunched beams in the absence of vacuum chamber shielding, see [4] and references therein.

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