LONG RANGE BEAM BEAM: TOWARDS FASTER COMPUTATIONS*

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Abstract
We outline some features of a program of study toward faster computation of the cumulative effect of a sequence of beam-beam interactions across the interaction region.

INTRODUCTION
The beam beam (BB) interaction between counter-rotating beams has a long history of study. The interactions may be long range (LR) or short range (SR) depending as the beams are separated or not (respectively). The interactions are impulsive, occurring each time bunches collide. Beam-beam limitations to the luminosity in proton-proton colliders has a long and relentless study for over four decades. Three machines have been the focus of these studies: the Tevatron and SCSC in the U.S. and the CERN LHC (and HL-LHC) in Europe.

The original intention was that this paper, with [1-3], would be the end goal in a concerted program to speed up the calculation of weak-strong beam-beam effects in the HL-LHC under the circumstances of unequal horizontal and vertical beam sigmas, asymmetric Interaction Region optics, and closed orbit distortions (COD).

PROGRAM OF STUDY
This program would calculate the short-range interaction at the central interaction point and the long-range (a.k.a. parasitic) interactions of the separated beams in the interaction region; and then sum these with the appropriate betatron phase advances; and then finally perform the sum over two interaction regions using Lie-algebraic techniques throughout.

For particle tracking studies we need the electric field and transverse impulse as function of transverse displacements. Analytic tune shift studies need the potential as function of action-angle coordinates. Analytic resonance studies need the Fourier components (with respect to angle) of the potential.

Each calculation is long, complicated and repetitive. It was hoped that significant speed up of the calculation could be made by exploiting several aspects of the problem.

1. Using a simpler beam density and interaction potential
2. Making a complete separation of the short-range and long-range interaction.
3. Exploiting symmetries that exist in the optics (such as phase advances, and equality of the ratios of beam sizes and separations) downstream and upstream of the central IP to introduce cancellations or simple summations of the up and down-stream interactions.
4. Use of pre-computed expressions for particular amplitudes such as 60, is useful to resonance studies – but not particle tracking.

Item (1) is the main area of novelty, and has already many ramifications: field, potential, tune-shifts, Fourier components, etc., all have to be re-calculated. Contrastingly, item (3), exploiting symmetries, is already performed by other authors for many years.

Currently, some parts of this program are complete and others well-started but not complete; and some parts have been the subject of diversions; for example, how to deal correctly with the potential function for the closed orbit distortion that results from the long-range interaction. Further, the calibration and comparison of new and old results has proven much more demanding than expected.

BB INTERACTION POTENTIAL
Particle tracking relies on computation of the electromagnetic impulses. Theory relies on analysis of the adiabatic invariants in presence of the BB potential. The impulse and potential for a single BB interaction are both complicated functions, usually developed in high order series. In a single interaction region (IR), there may be several short range and many long range BB encounters. In a ring, there may be several IRs. Consequently, many impulses (or potentials), which are individually complicated, must be calculated and added together. Therefore, short expansion series with equivalent accuracy will be useful.

Traditionally, the beams are taken to have Gaussian transverse charge distributions. In this model, the witness particle (in the weak beam) is always inside the strong beam (provided they are within the same physical aperture). This has the advantage for particle tracking, that no test is needed for inside versus outside.

Houssais and later Bassetti & Erskine, took the transverse density to be $G=\exp[-R^2/2]$ where $R^2 = \left((x/\sigma_x)^2+(y/\sigma_y)^2\right)$ and where unequal r.m.s. are $\sigma_x$ and $\sigma_y$. However, particle beams are not necessarily Gaussian unless synchrotron radiation is strong and the storage time is long. We take density in the quadratic form $Q = [1-(R/b)^2]^N$. With $b \geq 4\sigma$ and $N \geq 6$ suitably chosen, this can approximate the Gaussian increasingly well, even for relatively small $N\leq 6$. Let $V(G)$ and $V(Q)$ denote the potentials and $E(G)$ and $E(Q)$ denote the fields from the gaussian and quadratic density, respectively.

The power series expansions for field and potential are finite; and surprisingly accurate in the regime $b^2=(4+N)\sigma^2$. When we calculate the corresponding potential, for $R>4\sigma$ or larger, it differs little from the Gaussian case. This happens because the residual charge beyond radius

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>R is of order $\exp[-R^2/2]<<< 1$ leaving only the $\log[R]$ term, which is common to both potentials. In other words, at very large radius the only term that really counts in the field is $1/R$; the $E(G)$ and $E(Q)$ both get this dominant term correct (indeed for $R>b$, it’s the only term in $E(Q)$); but the series for $E(G)$ goes on to use many terms to correct for the charge at $R=b$.

**CONVERGENCE OF SERIES EXPANSION**

Concerning the cost of evaluating a mathematical function, we take the following model. The function has an argument which is evaluated once, and then we compute a power series in the argument – a series which has some convergence property. Typically, the larger is the argument, the greater is the number of terms that must be summed to obtain a relative fractional accuracy.

In the case of the $E(G)$, that series is in principle infinite whereas in the case of a power-law quadratic form $E(Q)$ the series is always finite. How could it be that they (gaussian and quadratic) give a similar result? The quadratic makes the simplifying assumption that all the charge is within radius $R=b$, whereas the gaussian has to worry about the residual charge beyond this radius even though it becomes progressively and vanishingly small.

**PARTITION OF LRBB**

The mathematical form of the long-range interaction is such that it contains also the short range interaction; it may be considered as the short range part (that does not contain the beam separation) and a residual “true long-range part” that contains terms in powers of the beam separation. These two parts have different symmetry with respect to reversal of the sign of the displacements, and so will add up differently either side of the IP. Figures 2a and 2b and the bullet points below attempt to explain that conclusion.

- The lattice functions ($\beta, \gamma$) are symmetric about the IP.
- Both beams share the same lattice functions ($\beta, \gamma$).
- For a single beam, ellipse tilt $\alpha$ is antisymmetric about the IP.
- The single-particle betatron phases flip by $\pi$ across the IP.
- The beam-beam force has the symmetry $F(-x,y,d) = F(x,y,d)$.
- Hence residual beam-beam forces do not cancel across the IP; instead they add.

These are important properties which result in simplifications.

**ASYMMETRIC INTERACTION REGION, UNEQUAL SIGMAS, AND COD**

The weak-strong model subjects the test particle in the weak beam to transverse impulses from the strong beam. The LRBB interaction is a function of the quadratic form: $[(x+Dx)^2/sx^2 + (y+Dy)^2/sy^2]$. Here $x,y = Sqrt[\beta_{weak}(s) x,y]Cos[\phi(s) x,y]$, $sx, sy = Sqrt[\beta_{strong}(s) x,y]$, and beam separations $Dx,Dy$ propagate the reference orbits away from the interaction point (IP) where $D_{x,y}=0$ and there are crossing angles. $Dx(s)$ is contributed to by the closed orbits of the weak and strong beams. To $1^{st}$ order $Dx = Dx_{weak(s)}+Dx_{strong(s)}$ proport $Sqrt[\beta_{weak}]Cos[...] + Sqrt[\beta_{strong}]Cos[...]$ where the cosine terms propagate the IP conditions. Unlike $x$ and $y$, $Dx,y$ do not advance in phase each turn. To $2^{nd}$ order, the constant terms in the LRBB impulse induce closed orbit distortions. These are compensated by impulses from magnetic elements $up/down$ stream of the IP.

In many cases, the interaction region [IR] is symmetric about the IP; and so $\beta_{[-s]} = \beta_{[+s]}$. Moreover, for round beams $\beta_x = \beta_y$. This results in substantial simplifications. Much of the BB literature adopts those...
simplifying conditions. However, this is not so for IR1 and IR5 of the Large hadron Collider (LHC). In this case, \( \beta_x \) not equal \( \beta_y \) and \( \beta(-s) \) not equal \( \beta(+s) \) except in the drift between the final quadrupole triplets. \( Dx=0 \) in IP1 and \( Dy=0 \) in IP5. All of this must be included in the appropriate symmetries.

We sketch the method with an abbreviated IR lattice (See Figure 3) which we take to extend from just downstream of D1L to just upstream of D1R.

Make a sequence of Lie maps; see Figure 4. The strategy is to move all beam-beam kicks to the IP @ 4. Insert identity transforms, then perform similarity transforms. Start at C3 and work left. Start again at C5 and work right. The result, see Figure 5, is a single equivalent element at the IP book-ended with linear optics.

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\begin{align*}
M(01) & \rightarrow A \rightarrow M(12) \rightarrow B \rightarrow M(23) \rightarrow C \rightarrow M(34) \rightarrow D \rightarrow M(45) \rightarrow C \rightarrow M(56) \rightarrow B \rightarrow M(67) \rightarrow A \rightarrow M(78) \\
& \leftarrow M14.M14^{-1} \leftarrow M24.M24^{-1} \leftarrow M34.M34^{-1} \leftarrow M45^{-1}.M45 \leftarrow M46^{-1}.M46 \leftarrow M47^{-1}.M47
\end{align*}
\]

Figure 4: Sequence of identity transforms are inserted and then similarity transforms (for the beam optics) enable the successive beam-beam interactions (labelled A-D-A) to be moved from the locations 1-8 to the central interaction point. Here M(nm) are optics, and (A,B,C,...) are residual beam-beam elements.

\[
\begin{align*}
M(04) & \rightarrow A(M14^{-1}z) \rightarrow B(M24^{-1}z) \rightarrow C(M34^{-1}z) \rightarrow D(z) \rightarrow C(M45z) \rightarrow B(M46z) \rightarrow A(M47z) \rightarrow M(48)
\end{align*}
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Figure 5: Result of applying the Lie maps is to concentrate an effective single beam-beam interaction that is the Lie product (shown as blue cells) at the IP book-ended with linear optics (shown as the pink cells).

REFERENCES

