# ANGULAR TRAJECTORY KICKS IN A HIGH-GAIN FREE-ELECTRON LASER

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*Abstract*

In a free-electron laser (FEL), transverse momentum offsets (or "kicks") are introduced either inadvertently (through wakefields or mis-steering of the electron beam) or as part of dedicated schemes that require off-axis radiation propagation [1]. Studying the influence of this effect on the performance of machines such as LCLS-I/II is critical both from a tolerance point of view and for its practical applications. A theoretical analysis of a high-gain FEL driven by such a "kicked" beam will be presented, with a critical evaluation of previous studies.

## INTRODUCTION

In a typical X-ray FEL configuration, a transverse kick (in other words, a discontinuity in the transverse slope of the electrons) leads to an oscillation of the electron beam about the undulator axis. Previous treatments of this effect in the context of FEL physics have focused on the case of a bunched electron beam, a hypothesis more relevant for the saturation regime [2]. Thus, the arguments used in the derivation of the basic analytical results are not quite applicable in the linear regime, where the buildup of the radiation power takes place. The object this paper is to highlight the results of a self-consistent analysis based on the linearized, Maxwell-Vlasov equations for an FEL driven by a "kicked" e-beam.

## THEORY OUTLINE

We begin by assuming that an (initially on-axis) electron beam receives a horizontal kick of magnitude $p_0$ at the entrance of the undulator module (designated as $z = 0$). Using the linearized Maxwell-Vlasov equations of the FEL, we can provide a complete description of the interaction between the radiation and the electron beam up to the onset of saturation [3]. Here, we summarize the most relevant results of our recent study [4]. The beam has a Gaussian profile in terms of transverse coordinates and energy and a flattop current profile with a bunch length considerably larger than the slippage length (so that a steady-state analysis is appropriate). For the planar undulator under consideration, the electric field of the linearly polarized radiation is $E_{\rm rad} = (1/2)E_\nu(\mathbf{x}, z)\exp(i\nu k_r(z-ct))$+c.c., where $E_\nu(\mathbf{x}, z)$ is a complex amplitude (which depends on the transverse position $\mathbf{x} = (x, y)$ and the longitudinal coordinate $z$) and $k_r$ is the resonant radiation wave number (c.c. stands for complex conjugate). Here, we focus on a specific frequency $\omega$, quantified by the scaled quantity $\nu = \omega/\omega_r$ ($\omega_r = ck_r$ is the resonant frequency).

In the latter stage of the linear regime, the amplitude of the radiation can be approximated by $E_\nu \propto [A_0(\hat{x}, \hat{y}) + A_1(\hat{x}, \hat{y})e^{i\hat{k}_{\beta x}\hat{z}} + A_{-1}(\hat{x}, \hat{y})e^{-i\hat{k}_{\beta x}\hat{z}}]e^{i\hat{\mu}\hat{z}}$. Here, $(\hat{x}, \hat{y})$ are some scaled transverse coordinates (in a frame co-moving with the oscillating e-beam), $\hat{\mu}$ is a complex growth rate, $A_0(\hat{x}, \hat{y})$ and $A_{\pm 1}(\hat{x}, \hat{y})$ are the mode profiles and $\hat{z} = 2\rho k_u z$ (where $\rho$ is the FEL parameter [5] and $k_u = 2\pi/\lambda_u$, with $\lambda_u$ being the undulator period). Moreover, $\hat{k}_{\beta x} = k_{\beta x}/(2\rho k_u)$ is a scaled focusing parameter, where $k_{\beta x} = 1/\beta_{ex}$ and $\beta_{ex}$ is the horizontal beta function (assuming the focusing lattice can be described by the smooth approximation). For future reference, we note that the corresponding focusing parameter for the $y$ direction is $\hat{k}_{\beta y} = k_{\beta y}/(2\rho k_u)$, with $k_{\beta y} = 1/\beta_{ey}$. Given the growth rate $\hat{\mu}$, the power gain length is $L_G = L_0\sqrt{3}/(2|\hat{\mu}_i|)$, where the index $i$ stands for the imaginary part and $L_0 = \lambda_u/(4\pi\sqrt{3}\rho)$. As far as the important $\rho$ parameter itself is concerned, it is given by

$$\rho = \left(\frac{K_0^2[JJ]^2}{16\gamma_0^3 k_u^2\sigma_x\sigma_y}\frac{I_p}{I_A}\right)^{1/3}, \tag{1}$$

where $I_p$ is the peak current of the beam, $I_A \approx 17$ kA is the so-called Alfven current, $\gamma_0$ is the average electron relativistic factor, $\sigma_{x,y}$ are the electron beam sizes (see below for more), $K_0$ is the dimensionless undulator parameter and $[JJ] = J_0(K_0^2/(4+2K_0^2)) - J_1(K_0^2/(4+2K_0^2))$ (where $J_0$, $J_1$ are Bessel functions). We should also point out the resonance condition that relates the basic FEL quantities, namely $\lambda_r = \lambda_u(1+K_0^2/2)/(2\gamma_0^2)$, where $\lambda_r = 2\pi/k_r$ is the resonant wavelength.

For a small kick angle $p_0$, the growth rate can be expressed as $\hat{\mu} = \hat{\mu}^{(0)} + \hat{\mu}^{(2)}$, where $\hat{\mu}^{(0)}$ is the unperturbed value and $\hat{\mu}^{(2)}$ is a second order contribution proportional to $p_0^2$. The fundamental amplitude $A_0(\hat{x}, \hat{y})$ has both zeroth and second order contributions, while the satellite amplitudes $A_{\pm 1}(\hat{x}, \hat{y})$ are first order quantities (proportional to $p_0$). The unperturbed quantities $A_0^{(0)}(\hat{x}, \hat{y})$ and $\hat{\mu}^{(0)}$ can be determined using the standard variational techniques [6]. In particular, for a Gaussian trial solution of the form $A_0^{(0)}(\hat{x}, \hat{y}) = \exp(-a_x\hat{x}^2 - a_y\hat{y}^2)$, the variational dispersion relation is

$$F_0(\hat{\mu}^{(0)}, a_x, a_y) = \hat{\mu}^{(0)} + \eta_{dx}a_x + \eta_{dy}a_y + a_x^{1/2}a_y^{1/2} \tag{2}$$
$$\times \int_{-\infty}^{0} d\hat{\xi}\hat{\xi}\exp[i(\hat{\mu}^{(0)} - \hat{\nu})\hat{\xi} - \hat{\sigma}_\delta^2\hat{\xi}^2/2]D_{0x}^{-1/2}D_{0y}^{-1/2} = 0,$$

where

$$D_{0x} = \frac{1}{4}(1 + i\eta_{\varepsilon x}\hat{\xi})^2 + (1 + i\eta_{\varepsilon x}\hat{\xi})a_x + a_x^2\sin^2(\hat{k}_{\beta x}\hat{\xi}),$$
$$D_{0y} = \frac{1}{4}(1 + i\eta_{\varepsilon y}\hat{\xi})^2 + (1 + i\eta_{\varepsilon y}\hat{\xi})a_y + a_y^2\sin^2(\hat{k}_{\beta y}\hat{\xi}). \tag{3}$$

The solution is completed by the relations $\partial F_0/\partial a_x = 0$ and $\partial F_0/\partial a_y = 0$, which are due to the fact that the stationary growth rate satisfies $\partial\hat{\mu}^{(0)}/\partial a_x = \partial\hat{\mu}^{(0)}/\partial a_y = 0$. These three equations yield the fundamental growth rate $\hat{\mu}^{(0)}$ and the mode parameters $a_x$ and $a_y$ as functions of the detuning $\hat{\nu} = (\nu - 1)/(2\rho)$. The other scaled quantities that require explaining are as follows: given the rms energy spread $\sigma_\delta$, the rms beam sizes $\sigma_{x,y}$ and the rms angular spreads $\sigma'_{x,y}$ (the matching conditions being $\sigma'_{x,y} = \sigma_{x,y}k_{\beta x,\beta y}$), $\hat{\sigma}_\delta = \sigma_\delta/\rho$ is the scaled energy spread parameter, $\eta_{dx} = (4k_u k_r \rho\sigma_x^2)^{-1}$ and $\eta_{dy} = (4k_u k_r \rho\sigma_y^2)^{-1}$ are the diffraction coefficients and $\eta_{\varepsilon x} = k_r\sigma_x'^2/(2\rho k_u)$ and $\eta_{\varepsilon y} = k_r\sigma_y'^2/(2\rho k_u)$ are the emittance parameters.

The two satellite amplitudes $A_{\pm 1}$ can also be determined using a variational technique, albeit a version that is suitable for driven (rather than homogeneous) equations. A suitable trial function for them is $A_{\pm 1}(\hat{x},\hat{y}) = \lambda_\pm \hat{x}\exp(-b_\pm\hat{x}^2)\exp(-c_\pm\hat{y}^2)$ and the corresponding variational function is

$$\begin{aligned} I_\pm = & \frac{\pi\lambda_\pm^2}{8b_\pm\sqrt{b_\pm}\sqrt{c_\pm}}[\hat{\mu}^{(0)} \pm \hat{k}_{\beta x} + 3\eta_{dx}b_\pm + \eta_{dy}c_\pm] \\ & + \frac{\pi\lambda_\pm^2}{8}\int_{-\infty}^0 d\hat{\xi}\hat{\xi}\exp[i(\hat{\mu}^{(0)} \pm \hat{k}_{\beta x} - \hat{\nu})\hat{\xi} - \hat{\sigma}_\delta^2\hat{\xi}^2/2] \\ & \times N_{1x}D_{1x}^{-3/2}D_{1y}^{-1/2} - i\pi\epsilon_1\lambda_\pm\frac{a_x}{(b_\pm + a_x)^{3/2}}\frac{1}{(c_\pm + a_y)^{1/2}} \\ & + \frac{i\pi\epsilon\lambda_\pm}{4}\int_{-\infty}^0 d\hat{\xi}\hat{\xi}^2\exp[i(\hat{\mu}^{(0)} - \hat{\nu})\hat{\xi} - \hat{\sigma}_\delta^2\hat{\xi}^2/2] \\ & \times N_x D_x^{-3/2}D_y^{-1/2}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} N_x =& a_x\exp(\pm i\hat{k}_{\beta x}\hat{\xi})\sin(\hat{k}_{\beta x}\hat{\xi}) \pm \frac{i}{2}(1 + i\eta_{\varepsilon x}\hat{\xi}), \\ D_x =& b_\pm a_x\sin^2(\hat{k}_{\beta x}\hat{\xi}) + \frac{1}{2}(1 + i\eta_{\varepsilon x}\hat{\xi})(b_\pm + a_x) \\ & + \frac{(1 + i\eta_{\varepsilon x}\hat{\xi})^2}{4}, \\ D_y =& c_\pm a_y\sin^2(\hat{k}_{\beta y}\hat{\xi}) + \frac{1}{2}(1 + i\eta_{\varepsilon y}\hat{\xi})(c_\pm + a_y) \\ & + \frac{(1 + i\eta_{\varepsilon y}\hat{\xi})^2}{4} \end{aligned} \tag{5}$$

and

$$N_{1x} = (1 + i\eta_{\varepsilon x}\hat{\xi})\cos(\hat{k}_{\beta x}\hat{\xi}), \tag{6}$$

$$D_{1x} = b_\pm^2\sin^2(\hat{k}_{\beta x}\hat{\xi}) + b_\pm(1 + i\eta_{\varepsilon x}\hat{\xi}) + \frac{1}{4}(1 + i\eta_{\varepsilon x}\hat{\xi})^2,$$

$$D_{1y} = c_\pm^2\sin^2(\hat{k}_{\beta y}\hat{\xi}) + c_\pm(1 + i\eta_{\varepsilon y}\hat{\xi}) + \frac{1}{4}(1 + i\eta_{\varepsilon y}\hat{\xi})^2.$$

In the relations given above, the only new scaled parameters are two quantities that are proportional to the kick angle $p_0$: $\epsilon_1 = p_0/(2\rho k_u\sigma_x)$ and $\epsilon = k_r p_0\sigma_x'/(2\rho k_u)$ (their ratio is related to the horizontal emittance $\epsilon_x = \sigma_x\sigma_x'$ since $\epsilon/\epsilon_1 = k_r\sigma_x\sigma_x'$). For this case, the variational relations are $\partial I_\pm/\partial\lambda_\pm = \partial I_\pm/\partial b_\pm = \partial I_\pm/\partial c_\pm = 0$. Solving these three equations yields the satellite mode parameters ($\lambda_\pm$, $b_\pm$ and $c_\pm$). In view of the simple quadratic dependence of $I_\pm$ on $\lambda_\pm$ and $\epsilon$, $\epsilon_1$, it can be readily seen that the resulting value for $\lambda_\pm$ is a linear combination of $\epsilon$ and $\epsilon_1$. Thus, as expected, $\lambda_\pm$ and $A_{\pm 1}$ are indeed first order quantities (recall that $\epsilon$ and $\epsilon_1$ are both proportional to $p_0$).

Having determined $\hat{\mu}^{(0)}$ and the parameters for $A_0^{(0)}$ and $A_{\pm 1}$, we can find the second order corrections to $\hat{\mu}$. The final analytical result is:

$$\begin{aligned} & [1 + ia_x^{1/2}a_y^{1/2}\int_{-\infty}^0 d\hat{\xi}\hat{\xi}^2 e^\Psi D_{0x}^{-1/2}D_{0y}^{-1/2}]\hat{\mu}^{(2)} = \\ & = -i\epsilon_1 a_x^{3/2}a_y^{1/2} \times [\frac{\lambda_+}{(a_x + b_+)^{3/2}(a_y + c_+)^{1/2}} \\ & + \frac{\lambda_-}{(a_x + b_-)^{3/2}(a_y + c_-)^{1/2}}] \\ & + \frac{\epsilon^2}{4}a_x^{1/2}a_y^{1/2}\int_{-\infty}^0 d\hat{\xi}\hat{\xi}^3 e^\Psi N_{0x}D_{0x}^{-3/2}D_{0y}^{-1/2} \\ & + \frac{i\epsilon_1^2}{4\eta_{dx}}a_x^{1/2}a_y^{1/2}\int_{-\infty}^0 d\hat{\xi}\hat{\xi}^2 e^\Psi D_{0x}^{-1/2}D_{0y}^{-1/2} \\ & + \frac{i}{4}\epsilon\lambda_+\sqrt{a_x}\sqrt{a_y}\int_{-\infty}^0 d\hat{\xi}\hat{\xi}^2 e^{\Psi_+}(\tilde{N}_x D_x^{-3/2}D_y^{-1/2})_+ \\ & + \frac{i}{4}\epsilon\lambda_-\sqrt{a_x}\sqrt{a_y}\int_{-\infty}^0 d\hat{\xi}\hat{\xi}^2 e^{\Psi_-}(\tilde{N}_x D_x^{-3/2}D_y^{-1/2})_-, \end{aligned} \tag{7}$$

where $\Psi = i(\hat{\mu}^{(0)} - \hat{\nu})\hat{\xi} - \hat{\sigma}_\delta^2\hat{\xi}^2/2$, $\Psi_\pm = \Psi \pm i\hat{k}_{\beta x}\hat{\xi}$, $N_{0x} = a_x + (1 + i\eta_{\varepsilon x}\hat{\xi})/2$ and

$$\tilde{N}_x = a_x\sin(\hat{k}_{\beta x}\hat{\xi}) + \frac{1}{2}[\sin(\hat{k}_{\beta x}\hat{\xi}) \mp i\cos(\hat{k}_{\beta x}\hat{\xi})](1 + i\eta_{\varepsilon x}\hat{\xi}).$$

This results completes our summary of the perturbation solution for the growth rate. Using these algorithms, we can calculate the full growth rate $\hat{\mu}$ as a function of the detuning $\hat{\nu}$ for different kick angles.

## NUMERICAL ILLUSTRATION

This section provides a brief numerical illustration of the theoretical technique outlined above. To begin with, we consider a set of LCLS-like FEL parameters involving the generation of 8.2 keV photons (0.15 nm radiation wavelength) with a 14.3 GeV e-beam and a 3 cm period undulator (with $K_0 = 3.7$). An average beta value of $\beta_{ex} = \beta_{ey} = 30$ m corresponds to a beam size $\sigma_x = \sigma_y = 23$ $\mu$m (for a transverse normalized emittance $\gamma_0\epsilon_x = \gamma_0\epsilon_y = 0.5$ mm-mrad). The energy spread is fixed at $\sigma_\delta = 10^{-4}$, while the e-beam peak current is at 3 kA, which yields an FEL parameter $\rho \approx 5 \times 10^{-4}$. In Fig. 1, we show the negative imaginary parts of the unperturbed growth rate $\hat{\mu}^{(0)}$ and its corrected counterpart $\hat{\mu} = \hat{\mu}^{(0)} + \hat{\mu}^{(2)}$ for a scaled detuning $\hat{\nu} = -0.5$ and a kick angle $p_0 = 1.54\mu$rad. These constant quantities are plotted against $z$ (scaled by the horizontal beta $\beta_{ex}$) in order to facilitate a comparison with the local, scaled growth rate $-\hat{\mu}_{zi}$. The latter is simply the logarithmic power growth




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rate $P_{\rm rad}^{-1} dP_{\rm rad}/dz$ (where $P_{\rm rad} \propto |E_\nu|^2$ is the radiation power) scaled by $4\rho k_u$. The necessary data for the radiation power are obtained from a solution of the linearized initial value problem (IVP) via an orthogonal expansion method [4]. After an initial transient stage, we observe a periodic evolution of the local growth rate with respect to $z$. Its average value agrees rather well with the figure obtained from the periodic analysis described in the theory section. This agreement encourages us to use the perturbed growth rate as a figure of merit for the FEL.

As a final task, we wish to compare the results of the periodic technique with previous analytical expressions for the kicked beam case. Of the latter, the main example is the Tanaka formula introduced in [2]. It states that a single kick $p_0$ increases the power gain length from an initial value of $L_{G0}$ to

$$L_G = \frac{L_{G0}}{1-(p_0/\theta_c)^2}, \tag{8}$$

where $\theta_c = \sqrt{\lambda_r/L_{G0}}$ is the critical angle. An equivalent statement would be that the ratio of the final (power) growth rate vs the initial growth rate is equal to $1-(p_0/\theta_c)^2$. In Fig. 2, we plot this growth rate ratio ($f_{\rm opt} = (\hat{\mu}_i^{(0)} + \hat{\mu}_i^{(2)})/\hat{\mu}_i^{(0)}$) vs the $\epsilon_1$ parameter (which is proportional to the kick angle) using both the formula and the periodic analysis. As calculated from the latter, the reduction of the growth rate due to the kick is stronger than that which is predicted by the formula. This observation can be conveniently quantified by rewriting the formula as

$$L_G = \frac{L_{G0}}{1-a_m(p_0/\theta_c)^2}, \tag{9}$$

where $a_m$ can be determined from the function $\hat{\mu} = \hat{\mu}(\hat{\nu})$. For our parameters, $a_m \approx 2$. This deviation should not be unexpected, given the different context of the two methods under consideration.

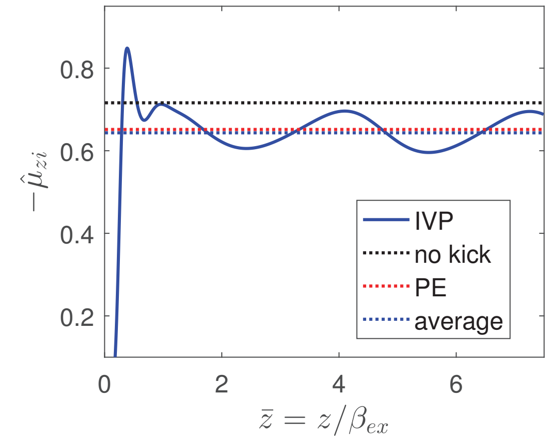


Figure 1: Evolution of the local power growth rate along the undulator. Also included are its average value, the unperturbed growth rate and the result of the periodic analysis.

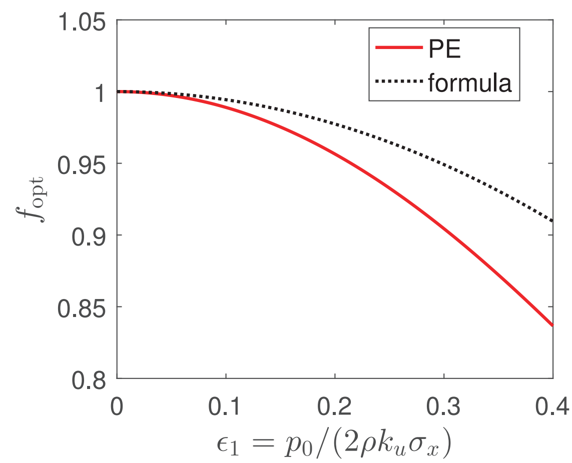


Figure 2: Ratio of the perturbed growth rate vs its unperturbed value (optimized with respect to the detuning $\hat{\nu}$) as a function of $\epsilon_1$. Values from the periodic analysis (PE) and the Tanaka formula are included.

## CONCLUSIONS

In this paper, we have outlined the results of a theoretical analysis of a high-gain FEL driven by a "kicked" beam. Working within the framework of a Maxwell-Vlasov formalism, we developed a periodic technique which allows us to define and determine an average FEL growth rate, suitable for describing the system after averaging over the e-beam centroid oscillations. After verifying its results through a comparison with other methods, the periodic analysis is contrasted with Tanaka's gain length formula, revealing a somewhat stronger quadratic reduction of the growth rate with respect to the kick.

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