HÉNON-HEILES SINGLE PARTICLE DYNAMICS AT IOTA

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Abstract

A Hénon-Heiles system is a simple, classical nonlinear Hamiltonian system offering a wide range of particle dynamics from regular orbits to resonant behavior to chaotic trajectories. Initially proposed to describe the motion of stars around a galactic center, it remains a vivid topic in Dynamics and Mathematical Physics since its discovery in 1964. Although the system and its modifications have been extensively studied numerically, its dynamics has never been observed in a controlled experiment. In this report we show that it is possible to create the Hénon-Heiles Hamiltonian using sextupoles in a realistic accelerator lattice. We propose a special sextupole channel to create the desired potential at the IOTA ring and study the 3D single particle dynamics by frequency map analysis and Poincare cross sections. The proposed experiment would allow real world testing of regular and chaotic motion with a controlled strength of the nonlinearity.

HÉNON-HEILES SYSTEM

In 1964, Michel Hénon and Carl Heiles [1] investigated the existence of the third isolating integral of galactic motion. Only two isolating integrals had been known – the total energy and the angular momentum. Nevertheless, observations of stars near the Sun and numerical computations of orbits in some cases behaved as if they had three isolating integrals. Through numerical computation the researchers searched for the third integral of motion in a model system. Not holding too hard to the astronomical meaning of the problem, the researchers only demanded that the potential they investigated was axially symmetric and the motion was confined to a plane. They found a model potential, which is analytically simple so that the orbits could be calculated rather easily but is still complicated enough so that the types of orbits are non-trivial. In Cartesian (x,y) coordinates this potential is written as

$$V(x,y)=\frac{1}{2}(x^2+y^2)+x^2y-\frac{1}{3}y^3 . \qquad (1)$$

It can be seen in Eq. (1) that the Hénon-Heiles potential is in fact composed of two harmonic oscillators coupled by a 3rd order term. The corresponding Hamiltonian is

$$H=\frac{1}{2}(p_x^2+p_y^2+x^2+y^2)+x^2y-\frac{1}{3}y^3=E \qquad (2)$$

where p_x and p_y are the momenta conjugate to x and y, respectively, and $E>0$ is the value of the Hamiltonian, which is conserved. The potential (1) has a finite escape energy of $E_{esc} = 1/6$. For values of energy $E < E_{esc}$, the equipotential curves of the system are closed and the motion is bounded. For $E > E_{esc}$, the equipotential curves open and three exit channels appear, through which the test particles may escape to infinity (Fig. 1).

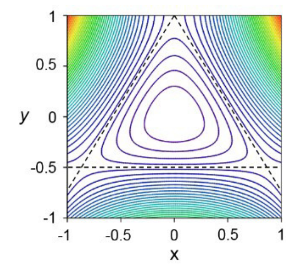

Figure 1: Equipotential lines of the Hénon-Heiles system (color). The motion is bounded for small initial amplitudes inside the separatrices $E = 1/6$ (black dashed).

Figure 2 depicts the Poincare maps of this system. For low energies ($E = 1/24$) the mapping plane is covered with the intersections of phase-space tori and the motion is regular. Above $E = 1/9$ most tori are destroyed and the map shows the coexistence of regular and irregular motion. Mostly irregular motion is observed close to the escape energy $E_{esc} = 1/6$.

The Hénon-Heiles model system showed that the galactic motion is integrable only for a limited set of initial conditions. Over the years, a lot of computational and analytical research has been devoted the system, but no experimental observations of its dynamics has been made so far.

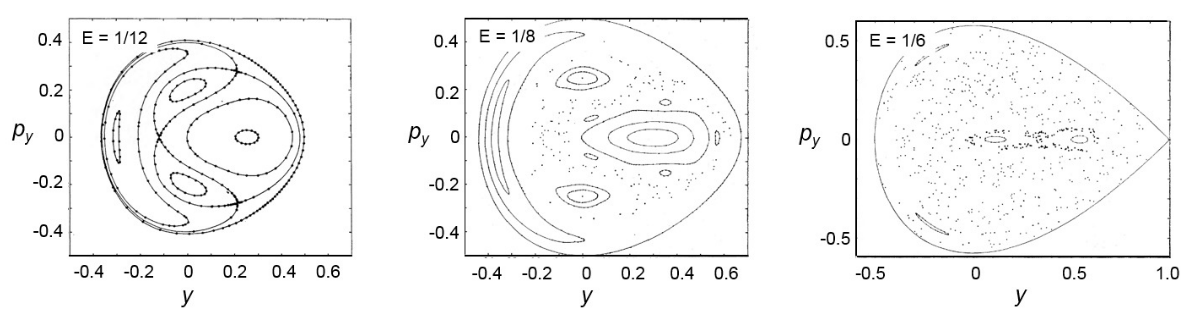

Figure 2: Poincare cross sections of the plane $x = 0$ for three values of parameter E: regular motion at $E = 1/24$, mix of regular and irregular motion at $E = 1/8$, and chaotic motion at $E = 1/6$. The particles are placed with the initial $y = 0$. The dots, which appear at random for E = 1/8 and E = 1/6, are generated by a single particle trajectory [1].

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ACCELERATOR IMPLEMENTATION

It is possible to create the potential (1) in a particle accelerator with the use of normal sextupole magnets. To achieve the time-independent Hamiltonial (2) one has to follow the approach, described in [2].

In an accelerator, the equations of motion in a transverse plane can be written as

$$z'' + K_z(s)z = 0 \quad K_z(s+C) = K_z(s), \tag{3}$$

where z stands for one of the two transverse coordinates x or y, s is the longitudinal coordinate, and $K(s)$ is the focusing strength, and C is the accelerator circumference. Here we assume that the coupling between the x and y degrees of freedom is negligible.

In the normalized phase-space coordinates:

$$z_N = z/\sqrt{\beta_z(s)} \quad p_{z,N} = p\sqrt{\beta_z(s)} - \beta'(s)z/2\sqrt{\beta_z(s)} \tag{4}$$

the initial time-dependent Hamiltonian, associated with Eq. (3), becomes time-independent in each plane:

$$H_{z,N} = \frac{1}{2}(p_{z,N}^2 + z_N^2) \tag{5}$$

The new "time" is the betatron phase advance, defined as

$$d\mu_z = ds/\beta_z(s) \tag{6}$$

and, in general, is different for the x and y planes. Because of that one cannot write a time-independent Hamiltonian of the 4D transverse motion if the beta-functions in x and y are different.

Now let us consider a straight section of an accelerator with $\beta_x = \beta_y = \beta(s)$ (the "time" flows equally fast in the two transverse planes) and a potential V, such that $\beta(s)V(x,y;s)$ is independent of the new time μ. The corresponding normalized Hamiltonian

$$H_N = \frac{1}{2}\sum_{z=x,y}(p_{z,N}^2 + z_N^2) + \beta \times V(x_N\sqrt{\beta}, y_N\sqrt{\beta}; s) \tag{7}$$

is autonomous. To obtain the Hénon-Heiles Hamiltonian (2) we need the potential to be cubic in x and y:

$$V(x,y;s) = \frac{\alpha}{\beta(s)^{5/2}}(xy^2 - \frac{1}{3}y^3), \tag{8}$$

where α is the strength parameter. This potential can be created by a set of sextupoles, powered in such a way, that their strengths are proportional to $\beta^{-5/2}$. The rest of the ring outside this section should have a linear transfer matrix of a thin axially symmetric lens with the phase advance of a multiple of 2π (Fig. 3).

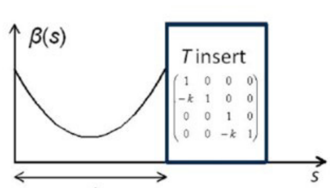

Figure 3: The ring outside of the sextupole channel section has a transfer matrix of a thin axially symmetric lens [2].

EXPERIMENT AT IOTA

A proof of principle experiment can be performed at the Integrable Optics Test Accelerator (IOTA) at Fermilab. IOTA is an electron storage ring, featuring a flexible linear lattice and a precise control of optics functions at the level of 10^{-3}. The ring has two 1.8 m long straight sections, specifically designed for the installation of nonlinear magnets, and the rest of the ring can be tuned to have a linear transfer matrix of a thin axially symmetric lens [3].

The proposed experiment will use the optics configuration with one nonlinear insert (Table 1). 18 sextupoles of the existing IOTA design can be installed in this section to create the potential (8) either equidistantly or with an equal phase advance between them (Fig. 4).

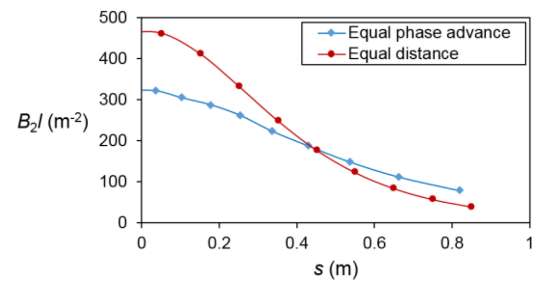

Figure 4: Installing the magnets with the equal phase advance between them leads to a lower variation of field strength than the equal spacing. Plotted is the integrated strength of the sextupoles as a function of the distance from the center of the channel. Total length 1.8 m, total phase advance – 0.3; a half of the channel is shown.

Poincare Cross Sections

IOTA will operate with a one-bunch pencil electron beam to test the dynamics in the ring. Shortly after the injection from the FAST linac, the transverse size of the bunch will shrink to its equilibrium value of ~ 0.1 mm thanks to the synchrotron radiation. IOTA's fast horizontal and vertical stripline kickers allow placing the electron bunch at an arbitrary point in the 4D transverse phase space. After the initial kick the evolution of particle trajectories can be studied using Poincare mapping technique. To create Poincare maps one can measure position of particles with 20 BPMs, distributed around the ring. The turn-by-turn position resolution of the order of ~ 1 µm should be attainable when using all the BPMs in the ring. Multi-turn data from the BMPs can be combined to reconstruct the Poincare map of the system using a Taylor series approach, proposed by Wang and Irvin [4].

To check whether the resolution of the BPMs will be sufficient to distinguish between the different types of orbits in the presence of field and lattice errors we performed a 4D tracking for 10^6 revolutions. For this study the ring outside of the section replaced with a linear transfer matrix. In the ideal case, with no errors, the one can clearly reconstruct the pictures of Poincare cross sections of the Hénon-Heiles system. With lattice errors at the level of 10^{-3} in betatron tune (design target for IOTA) and sextupole field quality of 5% some particles at high oscillation amplitudes are lost. Nevertheless, all trajectories remain bounded for 10^5 turns, allowing to clearly distinguish between the types of orbits (Fig. 5).

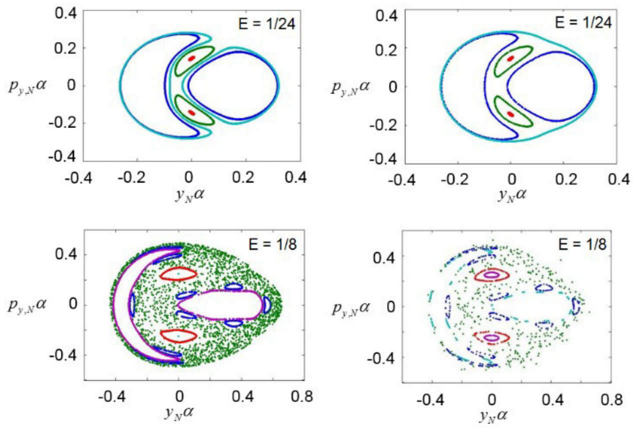

Figure 5: Poincare cross-sections of the plane $x = 0$ for the sextupole channel. Left – ideal case with no errors; right – with a random errors in magnet strength of 0.05 and the ring's phase advance of 10^{-3}. The particles are placed with the initial $y = 0$.

Particle Tracking and Frequency Map Analysis

Because the simplified Hamiltonian (7) does not take into account the longitudinal degree of freedom, we performed an accurate numerical tracking to check how it applies to a real accelerator. Our numerical model of the IOTA ring included dipoles with fringe fields, quadrupoles, and an RF cavity; the nonlinear potential (8) was created by a set of 10-cm-long hard-edge sextupoles. We simulated 3D particle dynamics in the accelerator with betatron and synchrotron motion coupled through dispersion and chromaticity using Lifetrac particle tracking code [5]. We used two methods to analyse the tracking results: dynamic aperture plots to find the region of the stable and motion, and Frequency Map Analysis [6] (FMA) to distinguish between regular and irregular motion.

Table 1: Main Parameters of IOTA Ring, Setup with One Nonlinear Insert [3]

Electron energy	150 MeV
Number of bunches, particles	1, 10^9
Ring circumference	40 m
Synchrotron radiation damping time	1 s
Equilibrium emittance, x & y, rms	0.04 mm-mrad
Betatron tunes, x & y	5.3, 5.3
Natural chromaticities, x & y	-11.4, -7.1
Synchrotron tune	5.3×10^{-4}
Harmonic number and voltage	4, 1 kV
Energy spread, rms	1.35×10^{-4}
Bunch length, rms	10.8 cm

As expected, at low amplitudes the particles executed regular motion. At higher amplitudes several resonant lines appear in the FMA. Some particles with the highest amplitudes escaped aperture probably due to the so-called Arnold diffusion [7]. The resulting aperture, measured on a time scale, comparable with the synchrotron damping time is of the ring is be about 70% of that of the ideal Hénon-Heiles system. The stable region encompasses areas of both regular, resonant, and chaotic motion (Fig. 6).

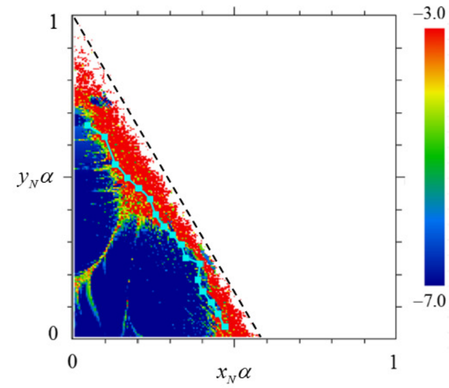

Figure 6: The dynamic aperture of the accelerator (light blue line) is about 70% of that of the ideal 2D system (black dashed line). Color denotes the FMA amplitude. Sextupole strength α = 800 $m^{-1/2}$.The aperture is calculated using the particle tracking for 10^5 turns; FMA – 2^{13} turns.

CONCLUSION

Hénon-Heiles system is a classic example of a time-independent Hamiltonian system that is both computationally simple and generic in its basic properties. The dynamics in the system is important for the research in the active field of nonlinear Hamiltonian systems.

We have shown that the Hénon-Heiles potential can be created and studied experimentally in a realistic accelerator setup, in particular, using a channel of 18 sextupole magnets in the IOTA ring. The required tolerances of 5% in sextupole field strength and 10^{-3} can be achieved in the IOTA. 6D tracking shows that under these tolerances the electron beam will remain stable for at least 10^5 revolutions. This will allow observing the coexistence of regular and irregular motion in the system. The resolution of the IOTA's 20 BPMs is sufficient to distinguish between different trajectories and reconstruct the Poincare cross sections of the phase scape.

Compared to the previous attempts to study nonlinear beam dynamics [8,9], the proposed approach allows precisely controlling the strength of the nonlinearity in the system. A similar measurement can also be performed at IOTA with an octupole channel, which is being built to show high achievable tune spread in a nonlinear focusing lattice [10].

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