Space Charge Induced Collective Modes and Beam Halo in Periodic Channels

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Outline

1. Review of study in collective mode.
2. Collective instability – structure resonance.
3. Numerical results of the unstable collective mode with TOPO code.
4. Halo mechanism – resonance between particle and collective mode.
5. Summary

In this talk we consider the space charge effect in rms matched beams only. Rms mismatched beams are beyond of the scope of this talk.
Collective mode - a long story to tell

- Real progress was made since the KV distribution beam was proposed (I. M. Kapchinskij and V. V. Vladimirskij, 1959)
- Rms envelope equation and the method of equivalent beam (Sacherer, 1971)
- Instability of oscillation modes of 2D constant focusing channel (R. L. Glustern, 1970)
  - Solve the perturbed Vlasov-Poisson equation simultaneously;
  - The perturbed potential is assumed as the form of hyper-geometric function in constant focusing channel.
- Instability of oscillation collective modes of 2D periodic focusing channel (I. Hofmann, et. al, 1983)
  - Solve perturbed Vlasov-Poisson equation simultaneously;
  - Perturbed potential is assumed as the form of polynomial for different orders collective mode;
  - Jacobi matrix of the perturbed Vlasov-Poisson equation is used to depict the stability.
- Instability of mismatched envelope oscillation (J. Struckmeier, et. al, 1984)
  - The envelope instability gives the same stop band as 2nd order even mode in ingo’s work (1983).
Recent discussion on collective mode: $2^{\text{nd}}$ order mode/$4^{\text{th}}$ order mode when $\sigma \sim 90$

- The balance between $2^{\text{nd}}$ and $4^{\text{th}}$ order collective mode was discussed at 2009.

- First measurement of a space charge structure resonance at GSI at 2009.


**Phys. Rev. Lett. 102, 234801 (2009)**
Recent discussion on collective mode: $2^{\text{nd}}$ order mode/$4^{\text{th}}$ order mode when $\sigma \sim 90$


- $2^{\text{nd}}$ order stop band is actually a coincidence of a $4^{\text{th}}$ order structure resonance.

- Similar phenomenon when $\sigma \sim 60$, Phys. Rev. Lett. 115, 204802 (2015).
Motivation of our work

• Clarify the mechanism of the collective mode analytically.

• Verify the effect of the collective modes numerically with self-consistent simulation.

• Show the effect of collective mode on particles – "beam halo".
2 Collective mode and structure resonance
Mathematical description of collective mode instability

Solve the perturbed Vlasov-Poisson equation simultaneously, self-consistent process

KV beam distribution:

\[ f_0 = \frac{N}{\pi^2} \delta(x^2 + p_x^2 + y^2 + p_y^2 - 1) \]

Perturbed KV beam distribution:

\[ f_1 = \delta f_0 = \frac{N}{\pi^2} \delta'(x^2 + p_x^2 + y^2 + p_y^2 - 1) \]

Perturbed KV Hamiltonian:

\[ H = H_0 + H_1 = \frac{1}{2\beta_x} (p_x^2 + x^2) + \frac{1}{2\beta_y} (p_y^2 + y^2) + \frac{1}{\epsilon} V(x, y, \ldots) \]

It has to meet

\[ f(x, y, z, p_x, p_y, p_z) = f(H) \quad \frac{df}{dt} = 0 \]

Thus, the perturbation of f and potential V has to satisfy:

\[ \frac{Df_1}{Ds} = \{ \frac{\partial}{\partial s} + \frac{1}{\beta_x} [p_x \frac{\partial}{\partial x} - x \frac{\partial}{\partial p_x}] + \frac{1}{\beta_y} [p_y \frac{\partial}{\partial y} - y \frac{\partial}{\partial p_y}] \} f_1 \]

\[ \Delta V = -\int f_1 dp \]

\[ = 2 \frac{N}{\pi^2 \epsilon} [p_x \frac{\partial V}{\partial x} + p_y \frac{\partial V}{\partial y}] \delta'(x^2 + p_x^2 + y^2 + p_y^2 - 1) \]

Mathematical description

Assuming the space charge potential perturbation inside

\[ V_i = \sum_{m=0}^{n} A_m(s)x^{n-m}y^m + \sum_{m=0}^{n-2} A_m^{(1)}(s)x^{n-m}y^m + \cdots \]

and space charge potential perturbation outside

\[ e^{-l(x-x_0)} \cos l\zeta, \quad e^{-l(x-x_0)} \sin l\zeta \]

With appropriate boundary condition

\[ \Delta \frac{\partial V}{\partial \xi} = \frac{Q}{\epsilon} \int \left( \frac{\partial}{\partial \psi_x} + \frac{\partial}{\partial \psi_y} \right) \left[ \cos \zeta \cos(\psi_x - \psi_x'), \sin \zeta \cos(\psi_y - \psi_y'); \right] ds' \]

\[ I_{j;k,l} = \int_0^s A_j(s') \sin[k(\psi_x - \psi_x') - l(\psi_y - \psi_y')] ds'; \]

\[ \frac{1}{C_{k,l}(s)} \frac{d}{ds} \left[ \frac{1}{C_{k,l}(s)} \frac{dI_{j;k,l}}{ds} \right] + I_{j;k,l} = -\frac{1}{C_{k,l}(s)} A_j. \]

\( I_{j;k,l} \): integral of the discontinuity of the surface electric field from period to period due to perturbation. The beam collective instability is decided by the Jacobi of \( I_{j;k,l} \).
Unstable mode: perturbation increases exponentially.

\[
\frac{1}{C_{k,l}(s)} \frac{d}{ds} \left[ \frac{1}{C_{k,l}(s)} \frac{dI_{j;k,l}}{ds} \right] + I_{j;k,l} = -\frac{1}{C_{k,l}(s)} A_j.
\]

\[
I_{j;k,l}(s + L) = M(L)I_{j;k,l}(s)
\]

With the form of Matthews equation, the stability of the system is decided by the eigenvalues of map \( M(L) \).

Physics mechanism of unstable mode – structure resonance!

- With the classic perturbation theory, the stability of perturbation is used to represent the stable characteristics of the whole system.

- The unstable collective mode lies in structure resonance, which is composed of *parametric resonance* and *confluent resonance*

\[
\Phi_{j;k,l} = n \times 180 \quad \text{Termed as parametric resonance}
\]

\[
\Phi_i + \Phi_j = n \times 360 \quad \text{Termed as confluent resonance}
\]

In the following numerical simulation, we mainly discuss the case where \( \sigma < 90 \). For simplicity, we use \( n\sigma \sim 180 \) to approximately represent the nth order of structure resonance.
3 Simulation results with TOPO code
Techniques used in TOPO:

• Under development;
• T-code;
• Moving cuboid meshes for space charge calculating;
• First weighting methods;
• Symplectic integrator;
• Poisson solver with FFT;
• Lorentz transformation (electric fields are multiplied by the factor of $1/\gamma^2$);
Numerical results with code TOPO

- Emittance growth ratio in 200 periods, eigenvalues of the 2nd, 3rd and the 4th order modes in FD and periodic solenoid channel when $\sigma_0 = 110$, KV initial beam.

- The broadened collective stop bands predict well the areas where the rms emittance growth take place.
mixed 2\textsuperscript{nd} / 4\textsuperscript{th} order stop band: $2\sigma \sim 180/4\sigma \sim 360$

\[ \sigma_0 = 110, \sigma \sim 80, \text{FD} \]

\[ V_2 = A_0(s)x^2 + A_2(s)y^2 \]

\[ \sigma_0 = 110, \sigma \sim 80, \text{FD} \]

\[ V_2 = A_0(s)x^2 + A_2(s)y^2 \]

\textbf{Emittance and phase advance evolution.}

- Both the 2\textsuperscript{nd} and 4\textsuperscript{th} order resonances are excited, in different time scales.
- Beam gets out of the stop band from the up-threshold, where space charge becomes less important.
- The beam gets a “local equilibrium” state finally, and it is a compromise between structure resonance and damping from density nonlinearity.
3rd order collective mode: $3\sigma \sim 180$

Stop bands pitches $\sigma_0 = 80$

Emittance and instantaneous phase advance evolution with initial KV beam $\sigma_0 = 80$, $\sigma = 50$.

$$V_3 = A_0 x^3 + A_2 xy^2 + A_0^{(1)} x$$

$\sigma_0 = 80$

$\sigma \sim 50$

FD

x-px phase space evolution along the FD channel, initial KV beam
$4^{th} / 3^{rd}$ order mode: $4\sigma \sim 180 / 3\sigma \sim 180$

Stop bands pitches $\sigma_0 = 80$

Emittance spilting!!!

Emittance and instantaneous phase advance

With initial KV beam, $\sigma_0 = 80$, $\sigma = 35$

$$V_4 = A_0 x^4 + A_2 x^2 y^2 + A_4 y^4 + A_0^{(1)} x^2 + A_2^{(1)} y^2$$

$\sigma_0 = 80$

$\sigma \sim 35$

FD

x-py phase space evolution along the FD channel
The $5^{th}$ order collective mode shows clearly if initial beam is KV beam, but damped out by the nonlinear density profile in WB beam case.

KV initial case has a larger emittance growth compared with the WB initial case—this is why higher order resonance are usually ignored.

Phase space profile, and emittance evolution curve for initial KV and WB distribution in FD channel.
Summary of the collective mode

• Show complex time dependent process and phase space transport behind the usual steady state presentation of this subject.

• Nonlinear space charge plays roles as resonance driving force and source of nonlinearity damping.

• Resonance will be self-detuned, beam will be self-adapted to an “equilibrium” state, where space charge will be less important—nonlinear saturation effect. The final beam would be a compromise of structure resonance and nonlinear damping.

• Higher order mode \( (n>4) \): could be easily damped out by the nonlinear profile distribution, that is the reason why they are usually ignored.

• “Beam halo” would be a side effect of the collective mode.
4 Interaction between particle motion and collective mode
Halo mechanism – resonance between particle and collective mode

The perturbed potential: \[ V_n = \sum_{m=0}^{n} A_m(s)x^{n-m}y^m \]

Single particle Hamiltonian: \[ H(x, p_x, y, p_y; s) = \frac{1}{2}(K_x(s)x^2 + p_x^2) + \frac{1}{2}(K_y(s)y^2 + p_y^2) + V_n(x, y; s) \]

Action – Angle frame: \[ H(\Phi_x, J_x, \Phi_y, J_y; \theta) = v_xJ_x + v_yJ_y + \sum_{p,k} A_{p,k}(\theta)\cos(p\Phi_x + k\Phi_y) \]

In Fourier expression: \[ V_n = \sum_{p,k} V_{p,k}(\theta) = \sum_{p,k} \sum_{l} G_{p,k;l}\epsilon^{i(p\phi_x + k\phi_y - l\theta + p\chi_x - k\chi_y)} \]

**Resonance condition: p+k=l**

The resonance condition is \( p+k=l \), related resonance strength is given by \( G_{p;k,l} \). In this case, perturbed space charge potential \( V_n \) is treated as external field. Resonance condition is the same as that in circular machine.
Halo mechanism – resonance between particle and collective mode

In the parametric resonance stop band, the perturbed mode also oscillating with a phase shift $\Phi^e$ in one period. 

$$\Phi^e = n \times 180$$

The perturbed potential is modified as

$$V_n = \sum_{p,k} V_{p,k} K(\theta) e^{i\Phi^e}$$

$$= \sum \sum \sum g_{p,k;l} e^{i(p\phi_x + k\phi_y - l\theta + \Phi^e + p\chi_x - k\chi_y)}$$

Resonance condition is modified as

$$p + k = l - 1/2$$, when \( n=1 \).
Halo mechanism – resonance between particle and collective mode

Example: 3\textsuperscript{rd} order resonance \( p=3, k=0, l=1 \)

\[
H(\Phi_x, J_x, \Phi_y, J_y; \theta) = \nu_x J_x + G_{3,0;l} \cos((3\nu_x - l)\theta + \xi_{3,0;l})
\]

Period 40, \( \sigma_0 = 80, \sigma \sim 50 \)

- In Cartesian and polar coordinate system for all particles at period 40. The black line in right figure represents the Hamiltonian tori obtained from analytical equation above. The black dots in the right figure represent the average action in each slice.
Halo mechanism – resonance between single particle and collective mode

Example: 4\textsuperscript{th} order resonance $p=4$, $k=0$, $l=1$

\[ H(\Phi_x, J_x, \Phi_y, J_y; \theta) = \nu_x J_x + G_{4,0;l} \cos(4\phi_x - l\theta + \Phi_{0;4,0} + \xi_{4,0;l}) \]

Period 20, $\sigma_0 = 80$, $\sigma \sim 35$
Halo mechanism – resonance between single particle and collective mode

“Beam halo”: particles located in the fold structures

Period 18, $\sigma_0 = 80, \sigma \sim 35$

- The interaction between particle and collective mode well describes beam behavior.

- Halo could be defined as the particle whose Hamiltonian beyond certain separatrix.

- Halo could be hidden in the “core” in one projected plane (x-px), but they will be shown in another plane (y-py).
Summary

- This talk only concentrates on nonlinearity from space charge in long period channel.
- Shows the complex time dependent process and phase space transport behind the usual steady state presentation.
- Resonance strongly depends on local parameters.
- Collective mode stop bands predict the area where the emittance growth take place quite well.
- If beam suffering from structure resonance, it will usually self-detuned into the safe region where the space charge will be less important.
- The final beam in a “local equilibrium” state is compromise of the structure resonance and damping from density nonlinearity.
- The higher order collective mode usually will be damped out.
- The unstable nth order collective mode is accompanied by an n-fold phase space structure.
- “Beam halo” could be grouped with particles whose actions are beyond the separatrix.
- Further study will be concentrated on collective modes in anisotropic beam in period focusing channel.
Thanks for your attentions!