FAST TRACKING OF NONLINEAR DYNAMICS IN THE ESS LINAC SIMULATOR VIA PARTICLE-COUNT INVARIANCE

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Abstract

Real-time beam modeling has been used in accelerator diagnostics for several decades. Along the way, the theory for matrix calculations of linear forces has matured, allowing for fast calculations of a beam's momentum and position distributions.

This formalism becomes complicated and ultimately breaks down with high-order beam elements like sextupoles. Such elements can be accurately modeled with a Lie-algebra approach, but these techniques are generally implemented in slower, offline multiparticle tracking software.

Here, we demonstrate an adaptation of the conventional Lie techniques for rapid first-order tracking of position, which is accomplished by treating a bunch's particle count as an invariant.

METHOD

In a previous paper, we outlined a framework for symplectic envelope-based tracking of non-linear forces using a Lie-algebra approach [1]. Here, we benchmark this method against an in-house multiparticle algorithm and a commercial multiparticle code (Tracewin [2]).

The formal conventions of the previous work have been retained and are briefly outlined here. Firstly, the Hamiltonian's position and momentum components are treated as a single phase-space vector:

$$\vec{v} = (q_1, q_2, \ldots q_n, p_1, p_2 \ldots p_n)^T \tag{1}$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{v} \equiv \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \cdot \partial_{\vec{v}}\mathcal{H} \tag{2}$$

where each I is an $n \times n$ identity matrix and $\partial_{\vec{v}}$ is the partial derivative with respect to each component of $\vec{v}$. For the solution of linear forces, this equation reduces to the symplectic transport matrices. For the non-linear forces of sextupoles or higher-order magnets, the Hamiltonian from Eq. (2) can be incorporated into a Lie algebra [3]:

$$\vec{v}_t = e^{-t:\mathcal{H}:}\vec{v}_i \tag{3}$$

Using nonrelativistic phase-space for the transverse components of a beam (where the time interval can substituted for a constant element length: $\Delta t \to L$) the exponential Taylor series takes form composed of recursive Poisson brackets:

$$\vec{v}_L = \vec{v}_i + [-L\mathcal{H}, \vec{v}_i] + \frac{1}{2!}[-L\mathcal{H}, [-L\mathcal{H}, \vec{v}_i]] + \ldots \tag{4}$$

Next, we can identify particle count as an invariant

$$N = \int_{\mathbb{R}^{2n}} \rho_{\vec{v}} d\vec{v} \tag{5}$$

which also remains fixed when splitting v into its constituents

$$\begin{aligned} N &= \int_{\mathbb{R}^n} \rho_{\vec{q}} \mathrm{d}q_1 \mathrm{d}q_2 \ldots \mathrm{d}q_n \\ &= \int_{\mathbb{R}^n} \rho_{\vec{p}} \mathrm{d}p_1 \mathrm{d}p_2 \ldots \mathrm{d}p_n \end{aligned} \tag{6}$$

where $\rho_{\vec{q}}$ and $\rho_{\vec{p}}$ are the respective position- and momentum-density components of phase space. Here, we are assuming that our output vector $\vec{v_t}$ can be decoupled so that $\vec{q_t} = f(\vec{q_i})$ and $\vec{p_t} = f(\vec{p_i})$, which will be addressed below.

To proceed, we can make use of Eq. 6 with the determinant of either component's Jacobian, which, using Eq. 3 for position in 1D, reduces to

$$|J_{x_L}| = \frac{dx_L}{dx_i} \tag{7}$$

where $q_i \to x_i$. This in turn yields

$$\rho_{x,L} = \frac{\rho_{x,i}}{|J_{x_L}|} \tag{8}$$

It is critical to note that $|J_{x_L}|$ needs an approximation for p in order to use Eq. 8—otherwise solutions for decoupled ρ_x and ρ_p in Eq. 6 cannot be assumed to exist. To achieve this, our envelope tests incorporated a Taylor approximation based on a trivial solution of the Courant–Snyder emittance equation:

$$p_i \approx -\frac{x\alpha}{\beta} - \frac{\sigma_x}{\beta} + \frac{x^2}{2\sigma_x\beta} + \frac{x^4}{8\beta\sigma_x^3} + \ldots \tag{9}$$

with an RMS profile width of $\sigma_x = \sqrt{\epsilon\beta}$ where ϵ is emittance. This approximation is only valid for $p \approx 0$. A similar approximation can be made for $x \approx 0$ to find $|J_p|$.

Flat initial proton distributions and exaggerated magnet lengths (~10 m) were used first to produce distinct profile shapes for easy comparison. Gaussian initial distributions were then tested for a more physical result. Tracewin simulations were initialized with energies of 750 MeV, null current (to neglect space-charge effects), and particle counts of ~300,000. To more easily match our field gradients with those of Tracewin, the multipole Hamiltonians used in Eq. 3 were simplified for 1D tracking as follows:

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$$\mathcal{H} = \frac{1}{n} k_n x^n + \frac{p^2}{2m} \tag{10}$$

where $n = 3, 4, 5$ for sextupolar, octupolar, and decapolar forces, respectively.[1]

RESULTS

Figures 1 and 2 show transverse profiles of a uniform proton bunch passing through octupolar and decapolar elements, respectively. In both cases, total counts are normalized to median bins. In Fig. 3, Gaussian initial distributions passing through long decapoles are shown (normalized to maximum-count bins).

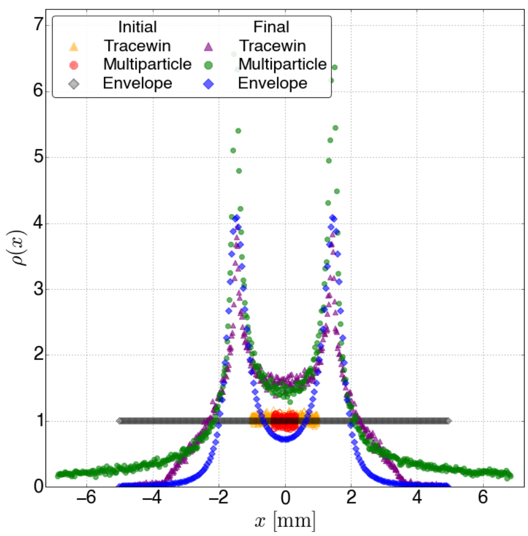

Figure 1: Uniform proton bunch profiles before and after passing through long octupoles. Shown are Tracewin, in-house multiparticle, and in-house envelope results for $\kappa = 0.18$ Wb^{-1}.

In all cases, α and β values were initially set to those reported by Tracewin. However, making fine corrections to these values proved useful in matching the envelope peak position and curvature to those of the multiparticle results. This dependence may have some utility in making high-precision corrections to Twiss parameters (e.g. populating an envelope every 5000 turns for a short multiparticle-based recalibration).

It should also be noted that for Fig. 1, a narrowed σ_x in the initial distribution of the multipole was required to prevent high ejection rates (which in this case did not affect the profile shape).

[1] Specific gradient strengths are reported here using Wille's definition [4], which allows for a cross-order normalization: $k_n \equiv (\frac{\delta}{\delta x})^{n+1} B$ [T m$^{-(n-1)}$], with field strengths normalized as $\kappa \approx \frac{e}{p} \frac{|k_n/\mathrm{T}|}{n}^{1/n}$ [Wb^{-1}]. Thus, for $p = 750$ [$\frac{\mathrm{MeV}}{c}$] and a quadupole of $B = 2.5$ T m^{-1}, $\kappa \approx 1$ [Wb^{-1}].

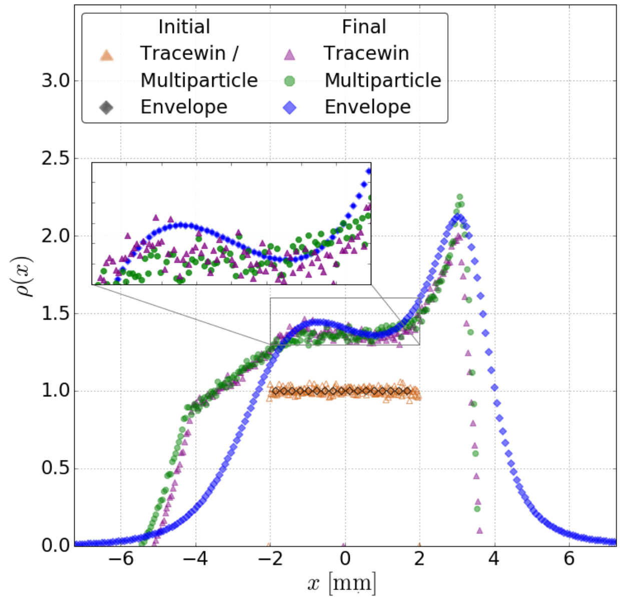

Figure 2: Uniform bunches passing through long decapoles. The inset highlights the strongly matched curvature around zero for the envelope and Tracewin results. $\kappa = 0.40$ Wb^{-1}.

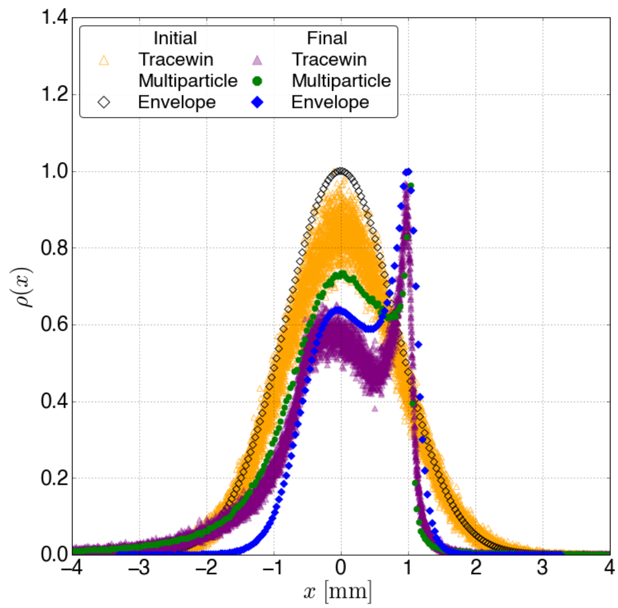

Figure 3: Gaussian bunches passing through long decapoles. A reduced magnet strength is used here to prevent higher losses owing to the Gaussian shape. $\kappa = 0.74$ Wb^{-1}.

Although Eq. 7 is explicitly symplectic, the in-house multiparticle results depend on the exponential Taylor series underlying Eq. 4. Because of this, the multiparticle results deviate from phase-volume preservation depending on the order of approximation. Thus, it was necessary to ensuring that the determinant of each Jacobian was approximately 1, that is:

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$$|J_{\vec{v}_L}| = \frac{dx_L}{dx_i}\frac{dp_L}{dp_i} - \frac{dx_L}{dp_i}\frac{dp_L}{dx_i} \approx 1 \quad (11)$$

in which all terms are nonzero, since any non-linear multipole Hamiltomian produces $x_L = f(x_i, p_i)$ and $p_L = g(x_i, p_i)$. For these tests, we used a cutoff of $|J_{\vec{v}_L}| = 1 \pm 10^{-5}$ for $x \approx \pm 2\sigma_x$.

To briefly address timing: since the envelope trials are independent of particle count, this method has an implicit speedup factor proportional to the N at which the shape of the multiparticle trial converges (usually forming accurate peaks near 10^4, with all features clearly defined around 10^6). From rudimentary timing trials, this corresponds to speedup factors of 3, 30, and 85, for N values of $10^5, 10^6$, and 10^7, respectively.

CONCLUSIONS

This method provides fast estimations of non-linear beamline forces of any order. As shown, its utility may be limited to use in linear-based codes as a periodic correction tool, albeit one of minimal computational cost compared with its multiparticle counterparts.

To develop fully symplectic non-linear envelope tracking, distributions in p and q can be carried explicitly through Lie transforms without relying on Eqs. 8 or 9. This can be accomplished using the substitution $\rho_{\vec{v}} \to \vec{v}$ in Eq. 3 and propagating the coupled distribution (e.g. a bivariate normal distribution).

Conversely, improvements on Eq. 9 may be possible by finding solutions for $x_L = f(x_0), p_L = g(p_0)$ that are not dependent on Twiss parameters.

Preliminary studies on both of these techniques are underway, and will also incorporate space-charge effects, a formalized truncation method, and cluster computing optimization.

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