DISTRIBUTED MATCHING SCHEME AND A DETERMINISTIC FLEXIBLE MATCHING ALGORITHM FOR ARBITRARY SYSTEMS*

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Abstract

Distributed matching scheme is explored, which can be advantageous over conventional schemes. In conjunction, a matching algorithm was developed for any configuration giving deterministic, rigorous solutions spanning trade-off between mismatch and quad strength. It shows promise of global optimum. With pre-calculated interpolation it further shows advantages of speed and flexibility.

DISTRIBUTED MATCHING

Matching in accelerators, where transverse beam profile or optical transport is brought in line with design using quadrupole and skew quadrupole magnets, is a key agenda in accelerator operation that motivated much algorithmic investigation [1]. Control of two important performance metrics, beam profile and orbit jitter, depends on the success of matching. The current report explores alternatives to conventional paradigm of matching, and introduces a supporting algorithm to enable this paradigm shift, with implication extending beyond matching.

Distributed vs Local Matching Configuration

Control of accelerator and beam properties follows two distinct paradigms in terms of geographical configuration, Distributed and Localized, often characterized and justified by cost vs performance:

Localized Control

- Limited/Costly/Bulky hardware (monitor and actuator)
- Little chance of cumulative/compounded error
- Damage mostly confined to local areas
- Example: Dispersion, Bunch length, Energy Spread

Distributed Control

- More affordable and compact monitors and actuators
- Errors accumulate & compound throughout entire line
- Damage arises everywhere and can be irreversible
- Example: Transverse orbit

Transverse matching shares almost all characteristics with parameters controlled in a distributed scheme, given the features of modern accelerator systems [2]. However, matching has historically acquired a local flavour where 100% of the matching must be accomplished locally, within a dedicated matching section of quadrupoles free of other interspersed elements. All quadrupoles outside the matching section only passively uphold the design transport once it is "matched" at a matching section, remote functionally if not geographically. Despite its broad and unquestioned acceptance, likely a relic from the era when computing power was inadequate to effect instant and global control of accelerators, this can lead to issues:

- Demand for dedicated single purpose matching sections imposes extra constraints on design.
- Cumulative mismatch causes problem everywhere, but is addressed only near matching sections.
- Cumulative mismatch that must be corrected locally causes excessive quad strengths.
- Excessive matching quad strengths cause beam blow-up, optical sensitivity, and irreversible nonlinear errors.
- With all matching concentrated in localized places, there is no recourse to algorithmic failure or configuration pathology (i.e., mismatch configuration not allowing a solution [1]). One is forced to live with the consequence.
- There is limited tolerance to errors in measurement and implementation, or provision for dynamic correction.

It is also worth noting that globally large beam envelope or jitter can result in sampling of nonlinearities in the transport optics in irreversible ways. Where exquisite elimination of aberrations is important, controlling envelope/jitter not only near matching sections is crucial.

The fact that matching has taken on a local flavour may be a legacy of traditional lack of algorithmic sophistication and computing power, much like orbit correction using all correctors that could fast degenerate into chaos before algorithm and computing power were up to the task. Fear of such chaos led to decision to keep most quads, or even correctors, off-limits for tuning in the absence of robust algorithms and real time global computer control. Unfortunately, this paradigm loses sight of the fact that in general, and particularly in long transfer lines, the primary function of all quadrupoles is envelope and jitter containment, and it is better done actively rather than passively, distributed across the entire line. So it would appear that the debate should not be about whether, but rather how, to use all quadrupoles for matching in an informed, systematic, and intelligent manner guided by competent algorithms, much like orbit correction using distributed steering magnets under such an algorithm.

RIGOROUS DETERMINISTIC MATCHING ALGORITHM

Trade-off between Objective and Constraint

By allowing the possibility of partial matching, a consequence of distributing matching over many sections, we transform the problem from absolute single-objective optimization to one of optimally balancing competing objectives. Matching is never a single-minded pursuit at the expense of all else, such as quad strengths, but neither has this trade-off been formalized and put on a rigorous framework, such as comes from the Lagrange multiplier

$$\begin{cases} \nabla F = \lambda \cdot \nabla H \\ H = h \end{cases} \rightarrow \; k_1^o, k_2^o, k_3^o, \ldots k_N^o, \lambda^o \rightarrow \; F = f(h)$$

where an objective function F of k_m to be optimized is subject to constraint function H of the same k_m. The

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solution is obtained by imposing the tangency condition with an arbitrary variable λ, and specifying the particular value of H=h. By scanning over h we get a complete picture of F played against H in a locally optimal sense everywhere. Equivalently, one can scan f, or even λ, to get an alternative view of the same trade-off [3].

$$\begin{cases}\nabla F = \lambda\cdot\nabla H \\ F = f\end{cases} \rightarrow H = h(f); \qquad \nabla F = \lambda\cdot\nabla H \rightarrow \begin{cases}F = f(\lambda)\\ H = h(\lambda)\end{cases}$$

The three formulations above for mapping out trade-off between objective and constraint can be shown to lead to differential relations between the optimal solution $\boldsymbol{k} = (k_1^O, k_2^O \ldots k_N^O)$ and f, h, λ or μ (=1/λ):

$$\left.\frac{d\boldsymbol{k}}{df}\right| = \frac{1}{\lambda}\cdot\frac{Adj(\boldsymbol{M})\cdot\boldsymbol{R}}{\boldsymbol{R^T}\cdot Adj(\boldsymbol{M})\cdot\boldsymbol{R}}, \qquad \left.\frac{d\boldsymbol{k}}{dh}\right| = \frac{Adj(\boldsymbol{M})\cdot\boldsymbol{R}}{\boldsymbol{R^T}\cdot Adj(\boldsymbol{M})\cdot\boldsymbol{R}}$$
$$\left.\frac{d\boldsymbol{k}}{d\lambda}\right| = \boldsymbol{M^{-1}}\cdot\boldsymbol{R}, \qquad \left.\frac{d\boldsymbol{k}}{d\mu}\right| = \boldsymbol{N^{-1}}\cdot\boldsymbol{S}$$
$$\boldsymbol{M_{ij}} = \frac{\partial^2(F(\boldsymbol{k}) - \lambda\cdot H(\boldsymbol{k}))}{\partial k_i\partial k_j}, \qquad \boldsymbol{R_i} = \frac{\partial H(\boldsymbol{k})}{\partial k_i} \qquad (1)$$
$$\boldsymbol{N_{ij}} = \frac{\partial^2\left(H(\boldsymbol{k}) - \mu\cdot F(\boldsymbol{k})\right)}{\partial k_i\partial k_j}, \qquad \boldsymbol{S_i} = \frac{\partial F(\boldsymbol{k})}{\partial k_i}$$
$$Adj(\boldsymbol{M}) = Cof(\boldsymbol{M})^{\boldsymbol{T}} = Det(\boldsymbol{M})\cdot\boldsymbol{M^{-1}}$$

where the vertical bar limits the derivative along the 1-dimensional curve of optimal trade-off. The above formulas make possible a deterministic program of integration between optimal constraint (μ=0) and optimal objective (λ=0), spanning all trade-off solutions in between. An equivalent but numerically more robust integration program deals with the direction cosines of $\boldsymbol{k}$:

$$\frac{d\boldsymbol{k}}{dk} = \pm\widehat{\boldsymbol{Q}}, \qquad \frac{d\mu}{dk} = \pm\frac{Det(\boldsymbol{N})}{|\boldsymbol{Q}|}, \qquad \boldsymbol{Q} = Adj(\boldsymbol{N})\cdot\boldsymbol{S}$$
$$\frac{d\boldsymbol{k}}{dk} = \pm\widehat{\boldsymbol{P}}, \qquad \frac{d\lambda}{dk} = \pm\frac{Det(\boldsymbol{M})}{|\boldsymbol{P}|}, \qquad \boldsymbol{P} = Adj(\boldsymbol{M})\cdot\boldsymbol{R} \qquad (2)$$
$$\frac{df}{dk} = \pm\boldsymbol{S^T}\cdot\widehat{\boldsymbol{Q}} = \pm\lambda\cdot\boldsymbol{R^T}\cdot\widehat{\boldsymbol{P}}$$
$$\frac{dh}{dk} = \pm\mu\cdot\boldsymbol{S^T}\cdot\widehat{\boldsymbol{Q}} = \pm\boldsymbol{R^T}\cdot\widehat{\boldsymbol{P}}$$

where we shorthanded $|dk|$ as dk, bold faced letters $\boldsymbol{k}$ and $\boldsymbol{P}$ denote vectors, and caret denotes unit vector. Integration starts with the first line at μ=0, continues until μ = λ = −1, then switches to the second line until the end, λ=0.

Algorithm Applied to Betatron Matching

The full 4D matching constrained by RMS quad strength off-design can be adapted into the above formulation with

$$F = \Phi = \frac{1}{4}\,Tr\left(\Sigma_D^{-1}\cdot M(k_m)\cdot\Sigma_R\cdot M^T(k_m)\right) \qquad (3)$$
$$\Sigma^{ij} = \frac{1}{n}\sum_{k=1}^{n} x_k^i\cdot x_k^j, \quad i,j = 1,2,3,4;\ m = 1,2,\ldots N_Q$$
$$H = \Delta\mathrm{K} = \sum_{m=1}^{N_Q}(k_m^R - k_m^D)^2 = \sum_{k=1}^{N_Q}\delta {k_m}^2$$

where Σ_D and Σ_R are the design and real beam covariance and M the transfer matrix of the N_Q-quad section that can encompass any intervening optics. F can be regarded as a 4D extension of the 2D mismatch factor [4]. This approach to matching displays many advantages over conventional methods based on nonlinear optimization or root-finding:

- Determinism – Guaranteed globally optimal solution
  - Deterministic start:
$$k_m = 0, \quad m = 1,2,\ldots N_Q \qquad (4)$$
$$\boldsymbol{N_{ij}}\big|_{\mu=0} = \frac{\partial^2 H(\boldsymbol{k})}{\partial k_i\partial k_j} = 2\delta_{ij} \rightarrow \left.\frac{d\boldsymbol{k}_i}{d\mu}\right|_{\mu=0,k_m=0} = \frac{1}{2}\left.\frac{\partial F(\boldsymbol{k})}{\partial k_i}\right|_{k_m=0}$$
  - No "inspired" initial guesses, no random search, no case-by-case parameter tweaking to guide the solution.
  - Deterministic end: If λ=0, Stop; If λ≠0, Don't Stop. Conventional algorithms can be ambivalent about the former, or stop short of the latter numerically and miss significant payoff, as shown in Figures 1 & 2.
- Works on any system, including XY-coupled and arbitrary interspersed optics within matching section.
- Computing demand is a slower function of increasing optics or system complexity than conventional methods.
- Global optima are deterministically mapped out and isolated (Pareto Front extraction) as shown in Figure 2.
- Complete range of options is given for optimal trade-off (Ideal for distributed matching).
- Points of diminished / enhanced return are identified by well-defined analytical procedure (**Det(*M*)**=0).
- Determinism, Robustness and Reproducibility are advantageous for interpolation or feedback applications.

The fact that this algorithm provides an entire range of options to optimally balance between Φ and ΔK makes it particularly suited for distributed matching. The ability to systematically identify points of diminished return further allows the user to logically pick the best such option. Each point above signifies a distinctive advantage over most of the conventional matching algorithms.

Restoring Determinism in Arbitrary Problems

The integration recipe is applicable not only to matching, but to any problem with a well-behaved analytical model. The determinism enjoyed by matching above, however, is due to known starting values for k_m in (4). The problem ceases to be deterministic if not even a single point on the trade-off curve is known a priori. This can fundamentally limit the usefulness of the recipe (1-2) applied to arbitrary problems. A program [3] nonetheless may overcome this and restore determinism. Assume the goal is to solve the trade-off between 2 functions F and G, neither of whose

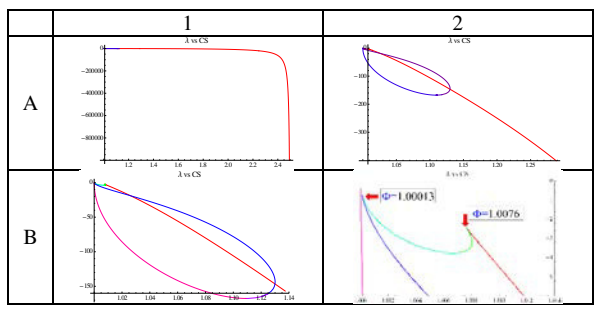

Figure 1. Solution path in the λ–Φ plane for a 6-quad 30° per cell FODO lattice. A1: Global solution path from $\lambda \rightarrow -\infty$ to λ=0. A2: Zoomed in for detail toward the end. B1: Further zoom. B2: Near absolute optimum (λ=0 and Φ=1): The path approached Φ=1 twice, indicated by arrows, at Φ=1.0076 & 1.00013, before executing a loop to Φ=1.14 (B1) and returning to Φ=1. Conventional algorithm can stop short of true optimum and return Φ=1.00013.

optimum, $\nabla F = 0$ or $\nabla G = 0$, is known. Notice that $\nabla F = 0$ only depends on F, thus a common terminus for trade-off curves between F and all other functions. Nothing prevents us from again taking a function with trivial starting point such as H of (3), integrating first from $\nabla H = 0$ to $\nabla F = 0$, then onwards to $\nabla G = 0$. Determinism is thus restored. This recipe may be exploited to maintain determinism in the algorithm (1-2) when applied to optimization of parameters beyond matching, even when no special case solution is known a priori.

Guiding Trade-Off Curve to Absolute Optimum

When used purely as a solution engine, the efficiency with which the algorithm homes in on the global optimum, $\nabla F = 0$, depends on a). Proximity of $\nabla F = 0$ to the starting point; b). Depth and steepness of $\nabla F = 0$ relative to lesser local optima, affecting how the trade-off curve is biased toward it. It is possible to improve the efficiency and robustness of the algorithm by artificially enhancing the slope of the objective $\boldsymbol{F}$, e.g. with an exponential amplifier, effectively distorting the topography of the solution space and biasing the trade-off curve toward the global optimum.

IMPLEMENTING DISTRIBUTED MATCHING

Segmentation of Matching Sections

In a distributed matching scheme the entire beam line is segmented into sections, all contributing to adiabatic reduction of mismatch Φ. The segmentation is flexible and needs not be contiguous. Special-purpose modules, such as RF or dispersion suppressor, can be left out or embedded inside a matching section, as indicated in Figure 3.

Interpolation on Pre-Calculated Trade-Off

The design Twiss Σ^D at the end of each section in Fig. 3 is a constant. This begs the question of why matching has to be computed repeatedly and likely haphazardly online.

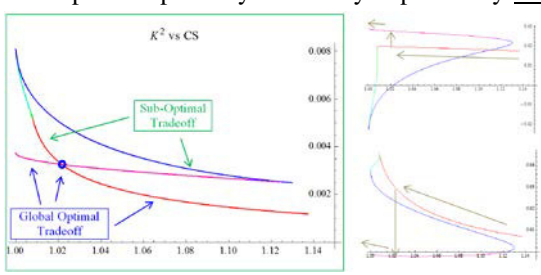

Figure 2. Left: Solution path in the ΔK–Φ plane roughly corresponding to B1 in Figure 1. By insisting on not stopping at Φ=1.00013 and pressing on to λ=0, we reached the true global optimum with a gain in ΔK by over 50%. This illustrates how a globally optimal trade-off curve, or Pareto Front, can be extracted by joining the red section with the magenta section at the intersection (blue circle), and discard everything above and to the right.
Right: Two quadrupole $\boldsymbol{k_m}$ vs Φ plots with the path of globally optimal trade-off indicated by gold arrows. The fact that λ is negative everywhere makes this extraction process unambiguous for both Φ and ΔK.

One can envision online matching via interpolation on offline-calculated solution tables constructed as functions of incoming mismatch amplitude and angle, plus control parameters of embedded modules (e.g. RF phase). This leads to considerable gain in speed, predictability, flexibility, and insight into the solution. Figure 4 gives an example. The matching algorithm used to generate such interpolation tables must be sufficiently deterministic and robust to rule out mass-scale case-by-case tweaking and to ensure smooth transition between interpolation vertices.

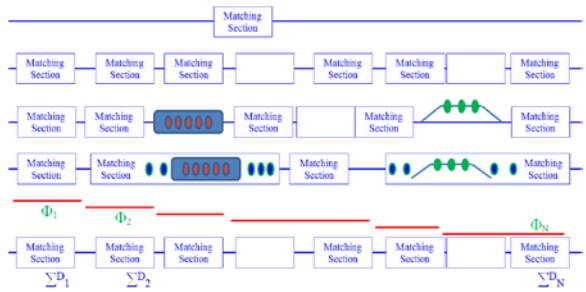

Figure 3: Concept of distributed matching. Top to bottom: (a) Localized matching (b) Segmentation into distributed matching sections, (c) Special modules left out, (d) Special modules embedded, (e) Adiabatic mismatch reduction by partially matching to design covariance Σ^D at each section.

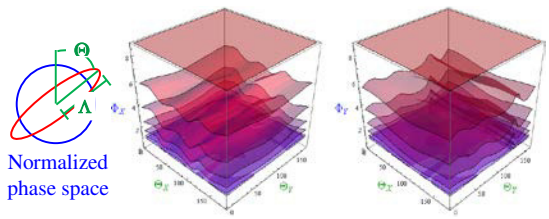

Figure 4: Distributed matching with solutions interpolated from offline table. Input mismatch is parametrized by amplitude Λ (related to Φ) and angle Θ in normalized space. Initial mismatch of $\Phi_{X/Y}$ =9 is launched, with $\Theta_{X/Y}$ covering entire range of 0–π (top sheet). The color sheets represent evolution of all initial $(\Phi, \Theta)_{X/Y}$ through 7 matching sections, reaching Φ=1 at the end (bottom sheet).

CONCLUSION

We discussed alternatives to conventional matching:

- Distributed scheme improving design flexibility, global beam envelope/amplitude, aberration suppression, quad strength, error tolerance, and dynamic correctability.
- Trade-off optimization leading to new matching algorithm with determinism, robustness and efficiency for more complex systems, global optimum isolation, and ability to analytically determine optimal trade-off points. The algorithm can be applied beyond matching.
- Interpolation on offline solutions, promising speed, predictability, and flexibility in online operation.

Possibility to extend the algorithm beyond matching, even when starting point is not known a priori, is discussed. The algorithm, a marked departure from convention, is the enabling factor behind all schemes discussed, besides being a powerful matching tool in its own right. The schemes can be modularly combined according to needs.

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